Modeling Preference in Non-Hausdorff Topological Spaces

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Outline

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Introduction Topological Approach to Preference:
Separability Axioms Not to be Taken for Granted

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Example 1 Non-Hausdorff Topology in Preference Aggregation

- Topology of utility space
- Preference (utility) aggregation and Chichilnisky's (1980) topological characterization of Arrow's impossibility theorem
- Non-Hausdorff topology of the utility space
- When is good aggregation possible? An investigation of 4 cases

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Example 2 Non-Hausdorff Topology in Preference Ordering

- Topology of binary relations
- Upper-set, Alexandrov topology, and specialization order
- Preorder (reflexive+transitive) versus strict partial order (irreflexive+antisymmetric+transitive)
- Classification of any pair of distinct points by T_0 and T_1

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An interval-scaled preference (utility) representation

An individual's preferences over d candidates is represented as a d-dimensional utility vector $\mathbf{u} \in \mathbf{R}^d$

This information is interval-scale, i.e. there is an equivalence relationship among utility vectors:

$$\mathbf{u} \sim \mathbf{v} \quad \text{iff} \quad \mathbf{u} = a \cdot \mathbf{v} + b \cdot \mathbf{1} \quad \text{for } a, b \in \mathbf{R}^1, a > 0$$

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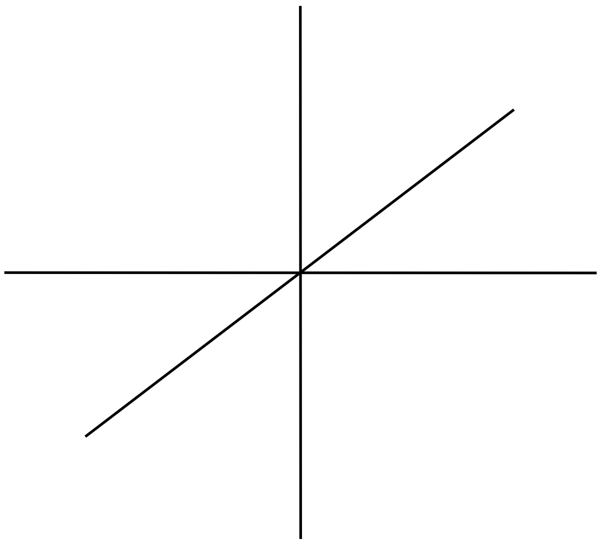

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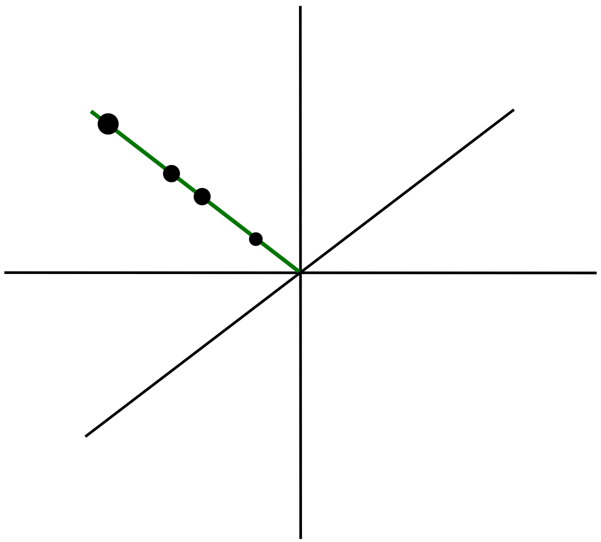

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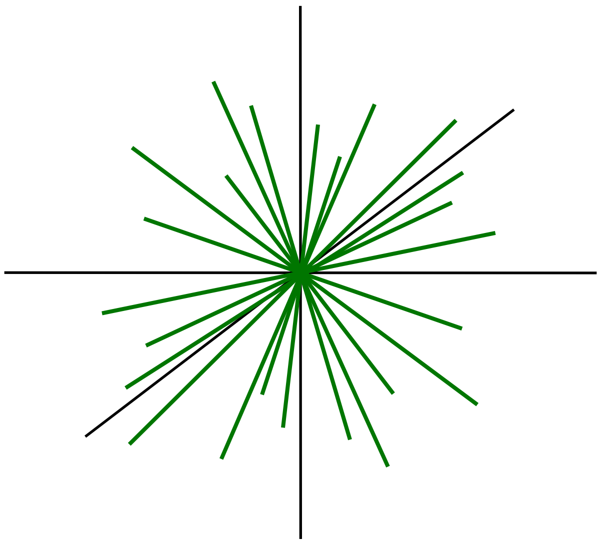

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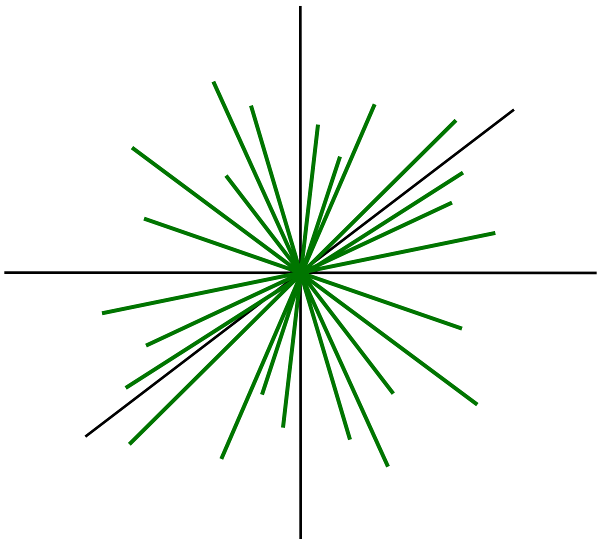

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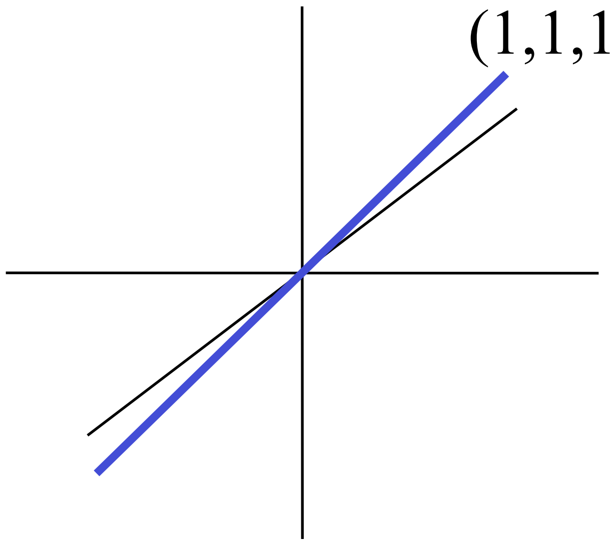

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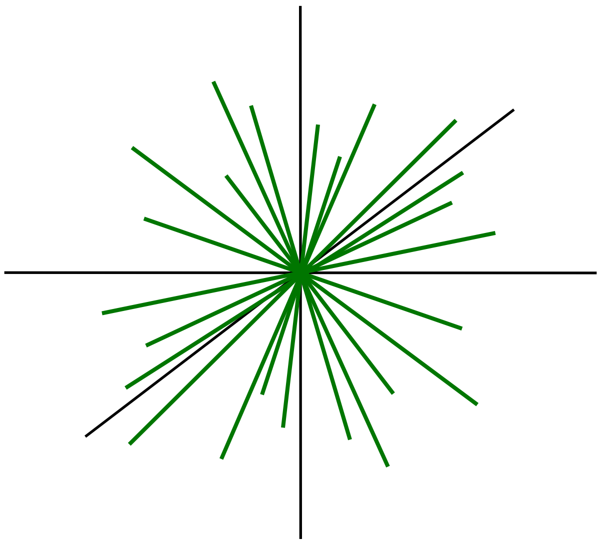

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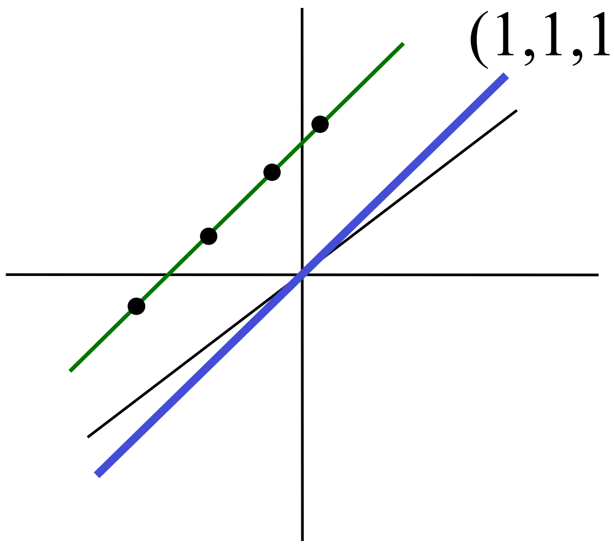

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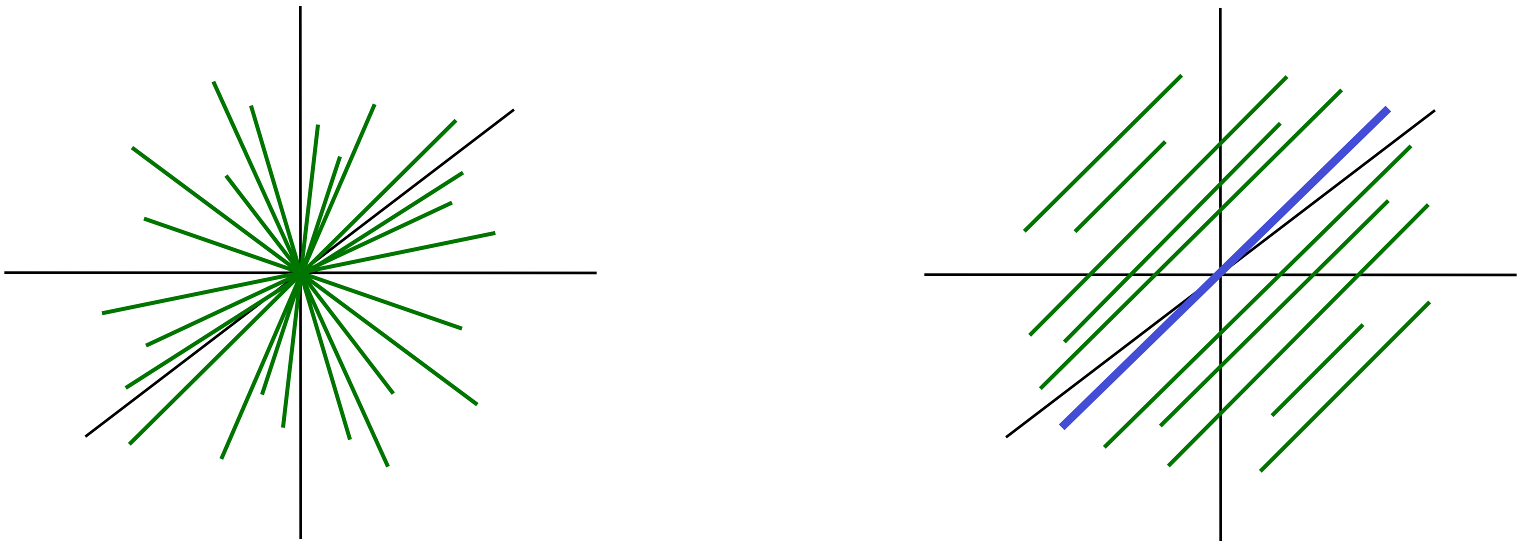

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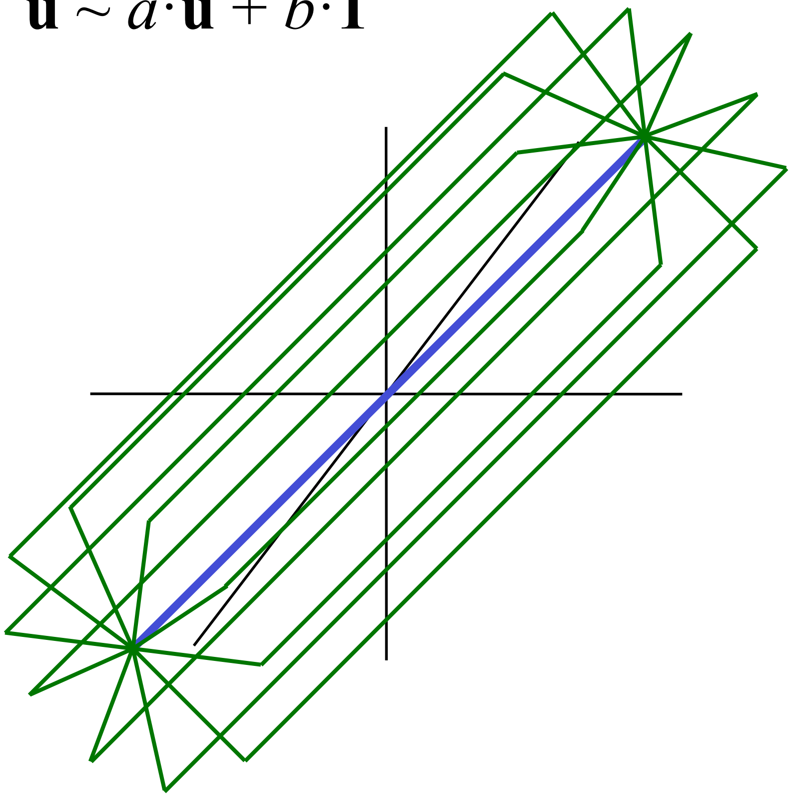

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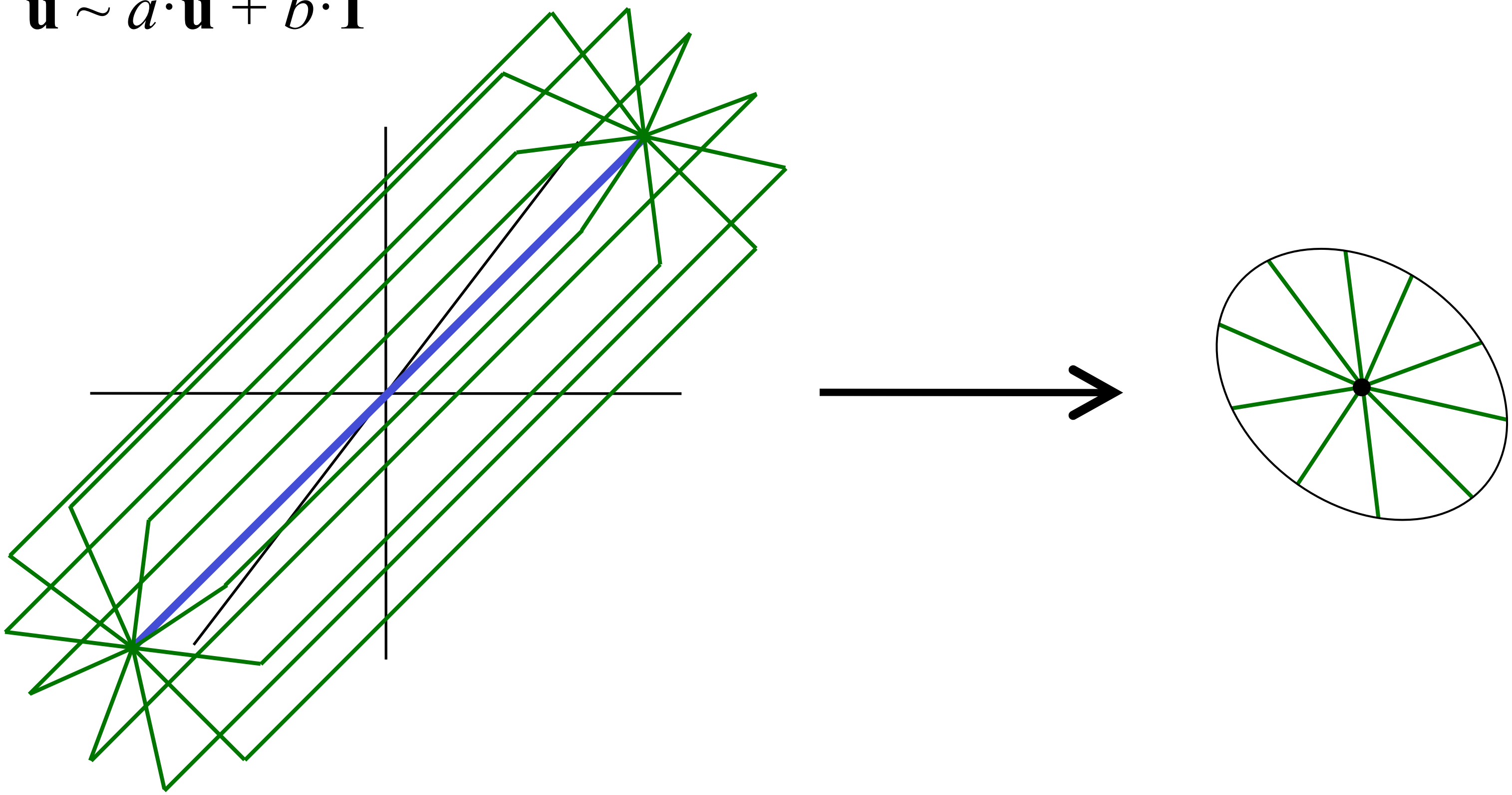

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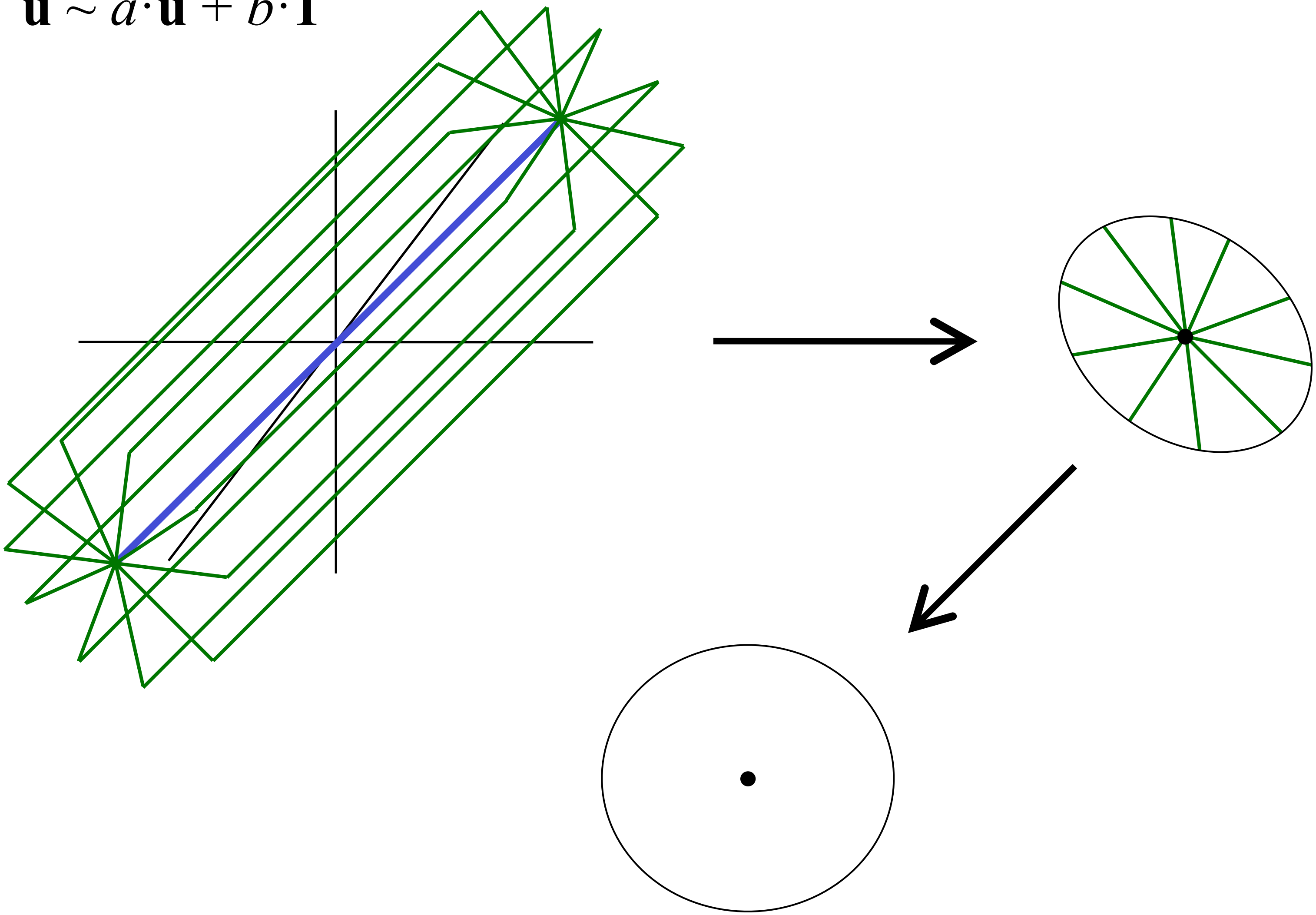
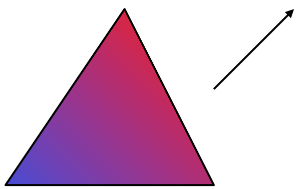


$$T^1 = \mathbf{R}^3 / \sim = S^1 \cup \{0\}$$

In general, preference space is $T^{d-2} = S^{d-2} \cup \{0\}$



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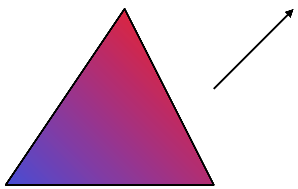
Space of probability distributions over *d* outcomes: Δ_{d-1}, *d*-1 dimensional simplex with *d* vertices.

Utility is a *valuation functional g*: $\Delta_{d-1} \rightarrow \mathbf{R}$.

Expected Utility theory ⇒ valuation functions *g* on Δ_{d-1} must be linear.



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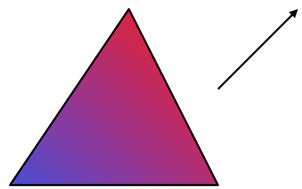

Individual preference may be represented by normalized gradient of valuation function on Δ_{d-1}, for non-constant g.

The space of normalized gradients (plus the constant map) is homeomorphic to T^{d-2}, as can be seen:

Taking g's gradient $\leftrightarrow$ $\mathbf{u} \sim \mathbf{u} + b \cdot \mathbf{1}$

Normalizing g $\leftrightarrow$ $\mathbf{u} \sim a \cdot \mathbf{u}$

How S^{d-2} is related to Borda scoring?

π: ranking (linear order) over d candidates; $\pi(i)$ gives the rank position of candidate i according to π

P_π : percentage of voters with ranking π, called "voters profile"

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and the convex mixture of them forms the combinatoric-geometric object "permutahedron".

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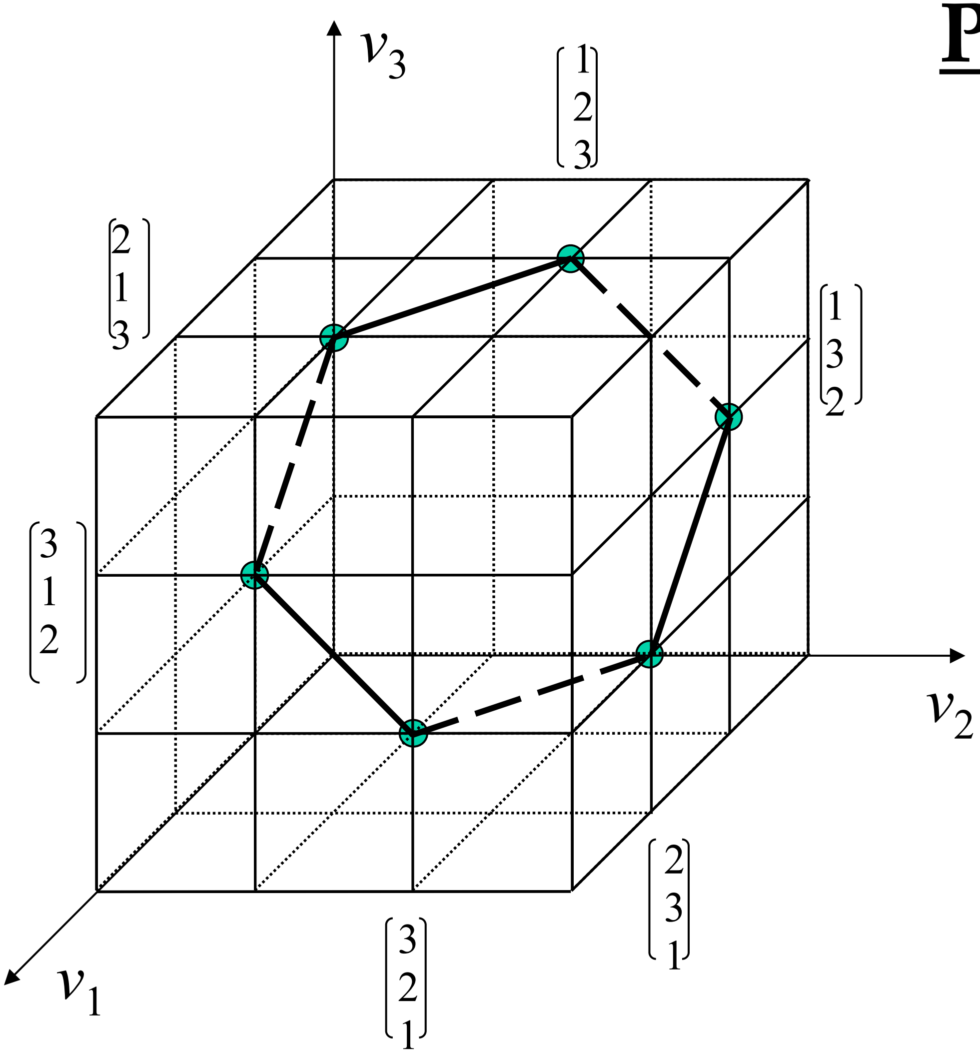

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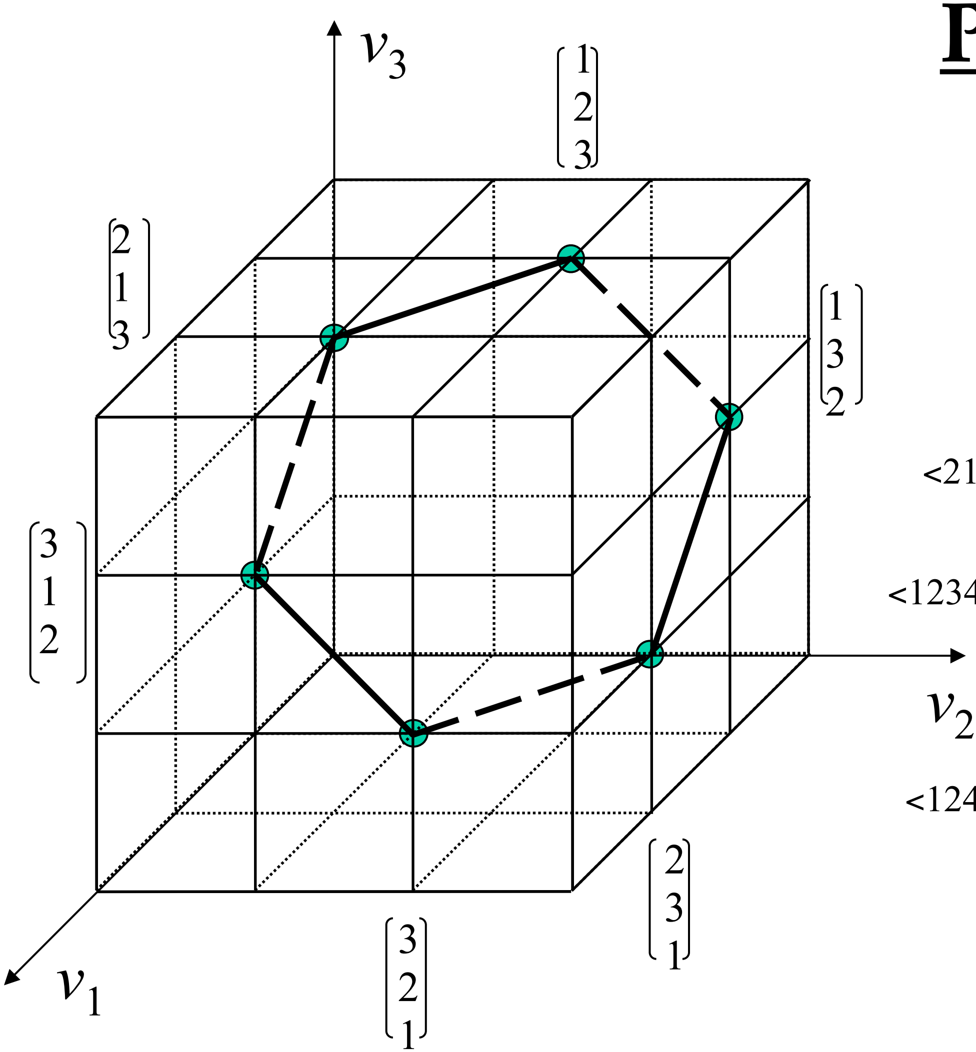

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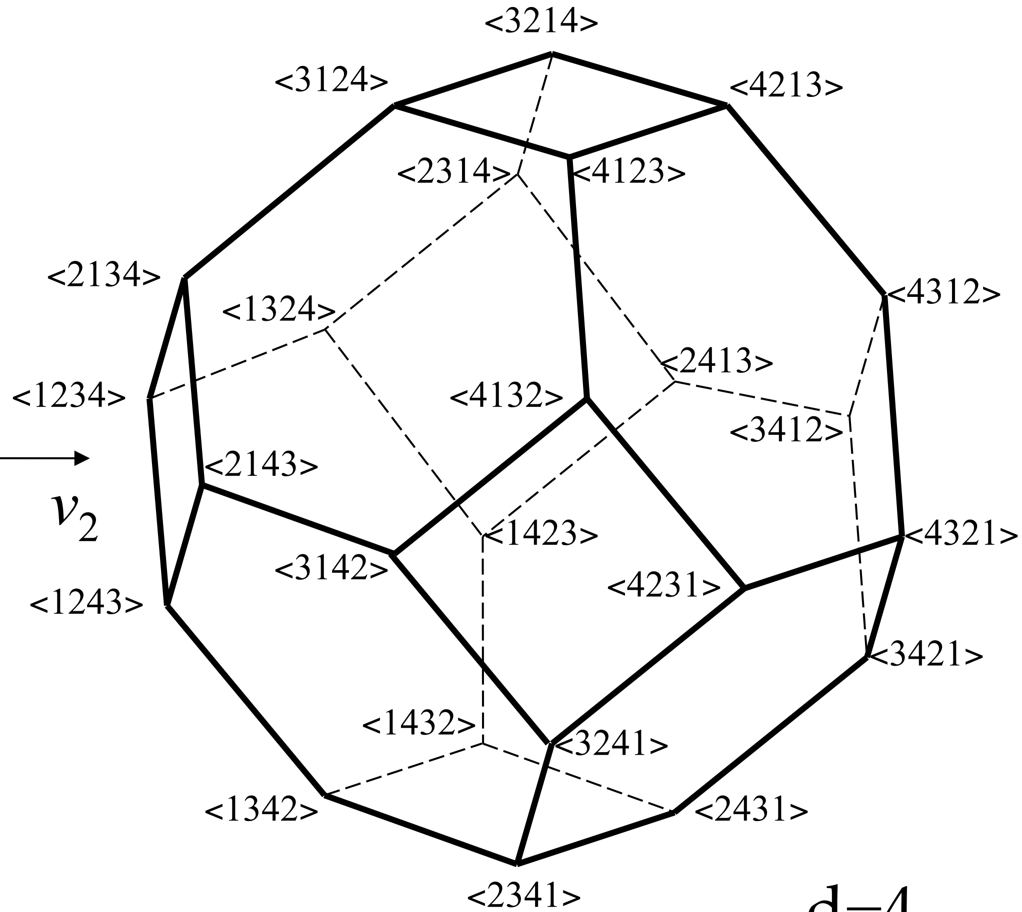

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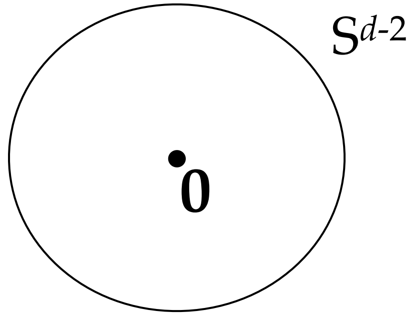
What is the proper topology for space T?



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- Space of individual preference over *d* candidates: $T^{d-2} = S^{d-2} \cup \{\mathbf{0}\}$

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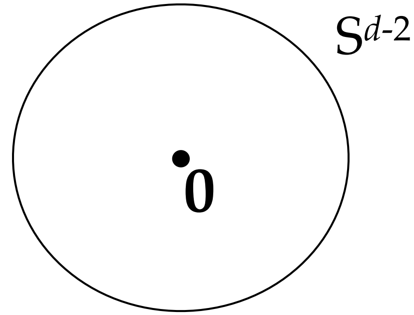

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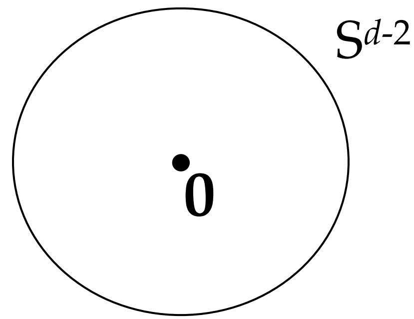

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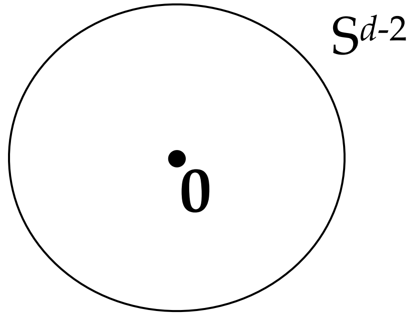

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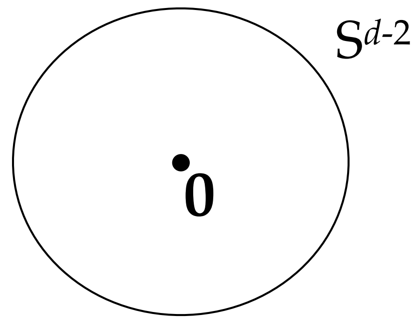

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- T is non-Hausdorff (T_0-separable only).

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- Non-contractibility of choice space, as CH showed, underlies various impossibility theorems including Arrow's (Baryshnikov, 1994).
- Previous work focused on S as choice space, i.e. with null preference excluded (treated as disconnected from S).
- The quotient topology $R^d/\sim$ on T makes the null preference a connected component and posits that the null preference is arbitrarily close to all other preferences.
- Treating $T = S \cup \{\mathbf{0}\}$ as connected space and endowing it with a non-trivial topology allows a refined analysis of the cause of impossibility theorem.

Properties of aggregation maps

- Continuity. Aggregated outcome depends continuously on voters' preferences
- Anonymity. Any permutation of the voters results in the same outcome
- Unanimity. $f(\underline{\mathbf{p}}) = p$
- Efficiency. $\mu(f^{-1}(\mathbf{0})) = 0$
- Pareto. If all voters prefer A to B, then outcome favors A over B. Geometrically: if $\forall i\ [p_i \cdot q \geq 0]$ then $f(\mathbf{p}) \cdot q \geq 0$
- WPA. If $f(\mathbf{p}) = -p_i$ then $f(-p_i, \ldots, -p_i, p_i, -p_i, \ldots, -p_i) \neq p_i$
- Dictatorship. $\exists j\ \forall \mathbf{p}\ [\mathrm{f}(\mathbf{p}) = p_j]$

Homotopy between aggregation maps: f_α for $\alpha \in [0,1]$; analog to continuous social change

Classification of aggregation maps

- Extensions from S to T for both individual and social choice to be investigated.
- Four possibilities:

Case I	$f: T^n = T\times\ldots\times T \to T$
Case II	$f: T^n = T\times\ldots\times T \to S$
Case III	$f: S^n = S\times\ldots\times S \to S$
Case IV	$f: S^n = S\times\ldots\times S \to T$

- Main challenge: dealing with non-Hausdorff topology of T and T^n

Case I

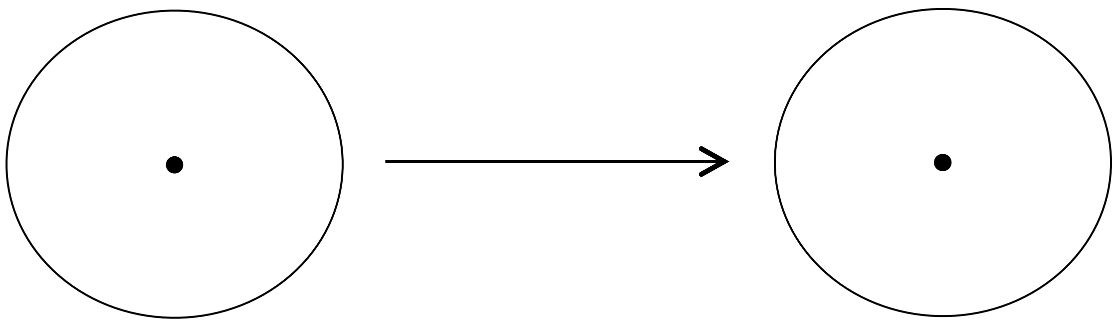
$$f : T^n \rightarrow T$$

(null preference allowed for individuals and for social outcome)

Given a voter j and profile $\mathbf{p}_{-j}$ for the remaining voters, consider the component map $f(\mathbf{p}_{-j},\text{-})$: $T \rightarrow T$.

This map represents j^{th} voter's effect on outcome.

$f(\mathbf{p}_{-j},\text{-})$:



T T

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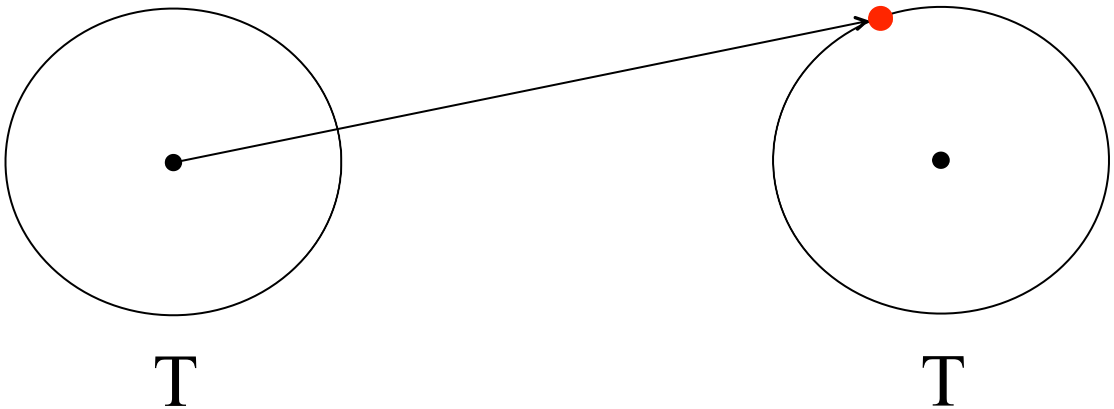

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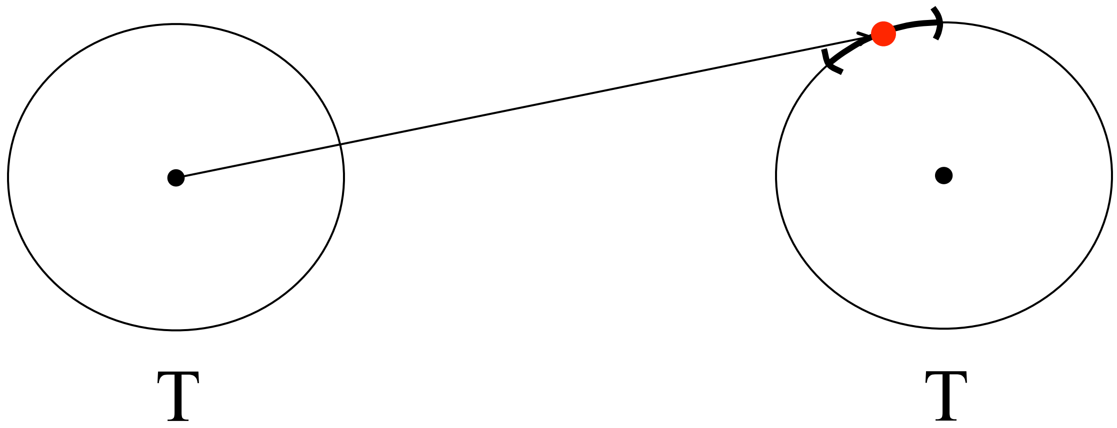

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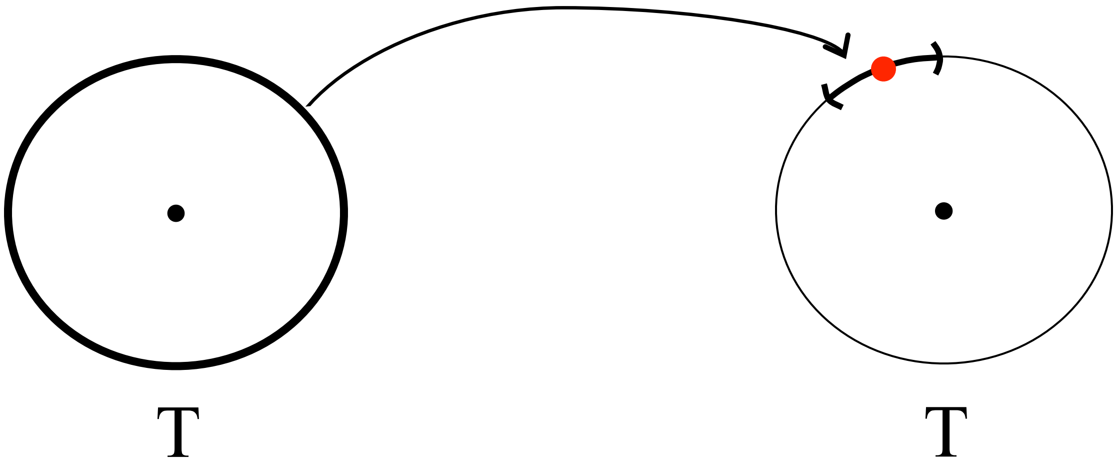

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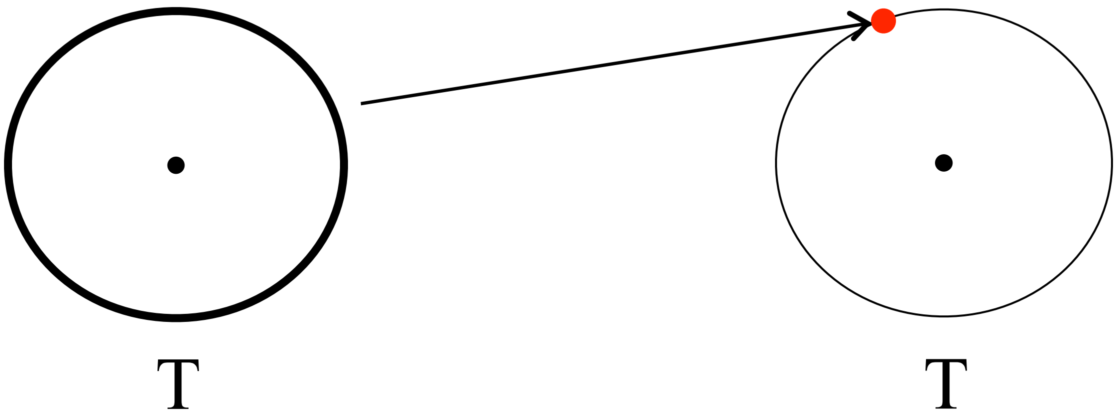


Conclusion: If $f(\mathbf{p}_{-j},0) \neq \mathbf{0}$, then $f(\mathbf{p}_{-j},\text{-})$ is constant.

$\Rightarrow$ In all situations, every voter either has the power to null the election (by voting **0**), or has no power at all.

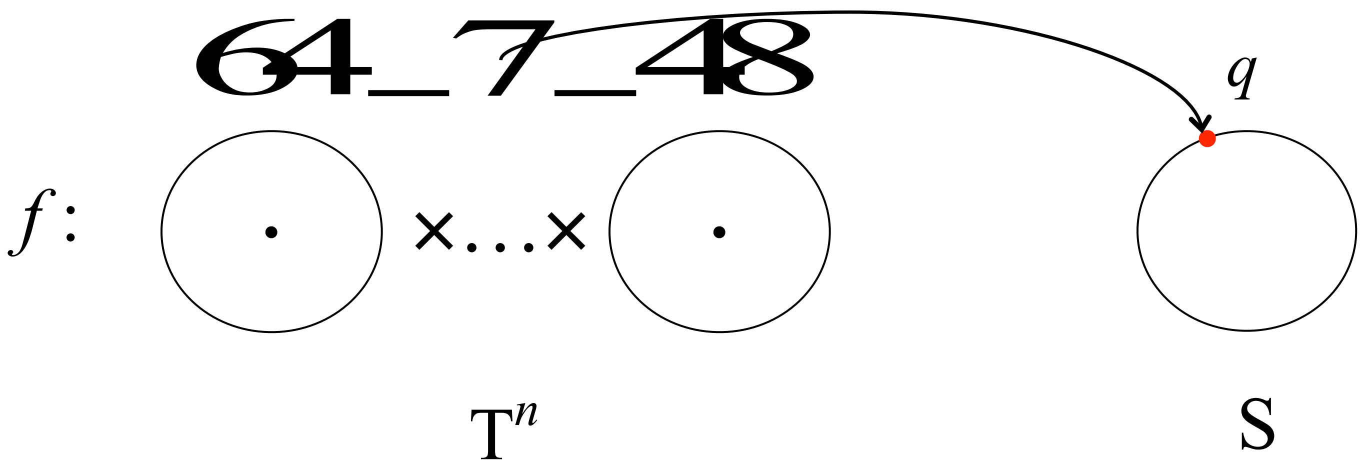
Case II

$$f : \mathrm{T}^n \to \mathrm{S}$$

(null preference allowed only for individuals)

In product topology of T^n, only open neighborhood of $(\mathbf{0}, \ldots, \mathbf{0})$ is T^n itself.
Since image space is S, $f(\mathbf{0},\ldots,\mathbf{0}) = q \neq \mathbf{0}$.
Using similar argument as from Case I, $\forall \mathbf{p}\ [f(\mathbf{p}) = q]$.

$\Rightarrow$ All continuous aggregation functions are constant.



Case III

$$f : S^n \to S$$

(null preference not allowed)

- Already treated in literature (Chichilnisky, 1980; Chichilnisky & Heal, 1983; Baryshnikov, 1994)
- Major results:
  - no continuous, anonymous, unanimous aggregation map (analogous to Arrow's Impossibility Theorem)
  - any Pareto map satisfying WPA is homotopic to dictatorship
- These negative results are motivation behind present extension of choice space S to T.

Case IV

$$f : \mathrm{S}^n \rightarrow \mathrm{T}$$

(null preference allowed only for social outcome)

- Possibility result: There can be continuous maps respecting anonymity, unanimity, efficiency, and/or Pareto.

  Example. "vector averaging map" (LeBreton & Uriarte, 1990):

$$f(\mathbf{p}) = \begin{cases} \sum_i p_i \Big/ \left\| \sum_i p_i \right\| & \sum_i p_i \neq 0 \\ 0 & \sum_i p_i = 0 \end{cases}$$

Case IV

All continuous maps homotopic, due to contractibility of image space T.

If f_0, f_1: $S^n \to T$ both continuous, anonymous, unanimous, efficient, & Pareto, define f_α: $S^n \to T$ for $\alpha \in [0, 1]$ by:

$$f_\alpha(\mathbf{p}) = \begin{cases} f_0(\mathbf{p}) & \alpha < \frac{1}{4} \\ \dfrac{(2-4\alpha)f_0(\mathbf{p}) + (4\alpha-1)f_1(\mathbf{p})}{\left\|(2-4\alpha)f_0(\mathbf{p}) + (4\alpha-1)f_1(\mathbf{p})\right\|} & \frac{1}{4} \le \alpha \le \frac{1}{2}, \phi(\mathbf{p}) < \delta \\ 0 & \frac{1}{4} \le \alpha \le \frac{1}{2}, \phi(\mathbf{p}) = \delta \\ f_0(\mathbf{p}) & \phi(\mathbf{p}) > \max\{\delta, \delta + (4\alpha-2)(M-\delta)\} \\ 0 & \phi(\mathbf{p}) = \delta + (4\alpha-2)(M-\delta), \frac{1}{2} \le \alpha \le \frac{3}{4} \\ f_1(\mathbf{p}) & \phi(\mathbf{p}) < \delta + (4\alpha-2)(M-\delta), \alpha \ge \frac{1}{2} \end{cases}$$

with $\phi(\mathbf{p}) = \sum_{(i,j)} \left\|p_i - p_j\right\|$, $M = \max_{\mathbf{p} \in S^n} \{\phi(\mathbf{p})\}$

and $\delta = \max\{d \mid \forall \mathbf{p} \ni \phi(p) \le d, \left\|f_0(\mathbf{p}) - f_1(\mathbf{p})\right\| < 1\}$

Summary of Example 1

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- As consequence, continuous aggregation maps f for four scenarios (S vs. T in individual and/or social choice)
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  - $f: S\times\ldots\times S \rightarrow T$ Exist anonymous, unanimous, efficient, Pareto maps

Extending image (S->T): enlarge space of maps, existence condition relaxed;
Extending domain (S^n->T^n): impose more restrictions on maps.

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- A combination of these properties leads to order relations
  - Preorder (reflexivity + transitivity)
  - Strict partial order (irreflexivity + antisymmetry + transitivity)
  - Equivalence (reflexivity + symmetry + transitivity)

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- Induced order and topology "reproduce" each other
  - P*'* is reflexive and transitive;
  - P*''* is irreflexive, anti-symmetric, and transitive

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- Note E and I are induced relations from P:
  - E is symmetric and transitive, and reflexive when P is -> "equivalence"
  - I is symmetric, and irreflexive when P is -> "intransitive indifference"

Semiorder, Interval order, and Weak order

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- Suitable topological spaces may only satisfy lower (i.e., non-Hausdorff) separability axioms.
- Need measure extension theorems on T_0 spaces (Keimel and Lawson, 2005).
- Combining topological approach to order theory (lattice of poset, semiorder and interval order, etc) and measure theory holds great promise.