Catastrophes, Rare Events, and Black Swans: Some Methodological Issues

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Beyond Monotonicity and Countably Additivity

Villegas, Arrow and Fishburn all introduced a monotonicity axiom so that Savage's characterization of subjective probability within an expected utility (EU) decision model could be extended to demonstrate existence of a countably additive subjective probability measure.

In several recent papers, Chichilnisky has explored a particular weakening of this monotonicity axiom.

It allows a revised decision theory in which rare events, catastrophes, perhaps even "black swans" can all be given more prominence.

It may be useful to contrast her approach with some alternative attempts to treat such issues.

Outline

Expected Utility

Economic catastrophes can be modeled as events so extreme that a suitable money metric utility function is undefined unless the probability of a catastrophe is sufficiently low.

Consider a consequence domain $K \times \mathbb{R}_+$ of pairs (κ, y) where:

1. $y \in \mathbb{R}_+$ is income or wealth (depending on context)
2. $\kappa \in K = \{0, 1\}$ is a binary variable indicating whether a "catastrophe":
   - occurs, iff $\kappa = 1$;
   - or does not occur, iff $\kappa = 0$.

Consider too a consumer whose preference ordering $\succsim$ on lotteries over $K \times R_+$ is represented by the expected value $\mathbb{E}u$ of a cardinal real-valued von Neumann–Morgenstern utility function $(\kappa, y) \mapsto u(\kappa, y)$.

Assumptions

The literature often regards $u(\kappa, y)$
as a state-dependent utility function of income y,
though it is a state-independent utility function
of a alertfully specified consequence (κ, y).

We assume that:

1. for each fixed $\kappa \in K$, the mapping $y \mapsto u(\kappa, y)$ is continuous, strictly increasing, and bounded above, with upper bound $\bar{u}_\kappa := \sup u(\kappa, y)$;
2. for each fixed $y \in \mathbb{R}_+$, one has $u(0, y) > u(1, y)$;
3. $\bar{u}_0 > \bar{u}_1$.

The second assumption, of course, is that the consumer
is worse off with a catastrophe than without.

It implies that $\bar{u}_0 \geq \bar{u}_1$, obviously,
but the third assumption strengthens this to a strict inequality.

Jacques Drèze and Michael Jones-Lee

Jacques Drèze (1962) "L'utilité sociale d'une vie humaine"
Revue Franąise de Recherche Opérationnelle, 23: 93–118.

Michael W. Jones-Lee (1974) "The Value of Changes
in the Probability of Death or Injury"
Journal of Political Economy 82: 835–849.

Jones-Lee (1974), following ideas due to Jacques Drèze (1962),
considers willingness to pay
for a reduced probability p of catastrophe.

Specifically, consider any reference or base line probability p^R
of a catastrophe, along with reference income levels y^R_κ
with and without a catastrophe.

Let $U^R := (1 - p^R)u(0, y^R_0) + p^R u(1, y^R_1)$
denote expected utility in the reference situation.

Money Metric Utility

One can use these reference levels and the equation

$$(1-p)u(0,m)+pu(1,y_1)=U^R$$

in an attempt to define implicitly
a money metric utility function $m(y_1;p)$.

This definition, when valid, implies that $m(y_1;p)-y_0^R$
is the consumer's willingness to accept
the net increase $p-p^R$ in the risk of catastrophe,
when compensation in the event of the catastrophe
rises from y_1^R to y_1.

Economic Catastrophes

The money metric utility function $m(y_1; p)$
really is defined by the equation

$$(1-p)u(0,m) + pu(1,y_1) = U^R$$

for the pair $(y_1; p)$ if and only if

$$(1-p)u(0,0) + pu(1,y_1) \leq U^R.$$

Otherwise giving up all income is insufficient to compensate
for the increase in p, which one could regard as a true catastrophe.

A Critical Probability Level

In particular, $m(y_1; p)$ is defined iff $p \leq p_C$, where

$$p_C := \frac{U^R - u(0,0)}{u(1, y_1) - u(0,0)}$$

is the critical probability above which no compensation
for any further increase in probability of catastrophe is possible.

It is perhaps worth noting that p_C, as the ratio of utility differences,
is preserved under positive affine utility transformations.

In fact, p_C equals the constant marginal rate of substitution
between shifts in probability away from $(0, 0)$,
the worst possible outcome without a catastrophe, toward:

1. the reference position;
2. the catastrophe combined with the income level y_1.

Extreme Economic Catastrophes

One can also have an extreme catastrophe where

$$(1-p)u(0,0)+p\bar{u}_1 > U^R$$

or

$$p > \frac{U^R - u(0,0)}{\bar{u}_1 - u(0,0)}$$

This implies that the probability of catastrophe is too high for compensation to be possible, no matter how high y_1 may be.

Outline

Rare Events and Infinitesimal Probabilities

Rare events can be modeled as having infinitesimal probability in an extended EU theory with “non-Archimedean” probabilities.

Their values lie in the metric space completion $R^*(\epsilon)$ of Lightstone and Robinson’s algebraic field $R(\epsilon)$, defined as the smallest that contains both the real line $\mathbb{R}$ and one basic positive infinitesimal ϵ.

A.H. Lightstone and A. Robinson (1975)
Nonarchimedean Fields and Asymptotic Expansions
(Amsterdam: North-Holland)

P.J. Hammond (1997, 1999) “Non-Archimedean Subjective Probabilities in Decision Theory and Games” Stanford University Dept. of Econ. Working Paper No. 97-038; abbreviated version in *Mathematical Social Sciences* 38: 139–156.

Standard Decision Theory

Standard decision theory uses the expected utility (EU) criterion.

Traditionally, moreover, a distinction is made between objective and subjective EU theory, depending on whether one faces:

- risk or roulette lotteries described by objective probabilities, as in von Neumann and Morgenstern (1944) and then Jensen (1969);
- uncertainty or horse lotteries described by subjective probabilities, as in Savage (1954);
- combinations of roulette and horse lotteries, as in Anscombe and Aumann (1963).

Infinitesimal Probability

Reinhard Selten (1975) "Re-examination of the Perfectness Concept for Equilibrium Points of Extensive Games" *International Journal of Game Theory* 4: 25–55.

To accommodate rare events, one can follow the game-theoretic literature emanating from Selten (1975).

This allows "trembles" with a probability that is some positive multiple of an infinitesimal ϵ, taken to be some entity that is smaller than any positive real number because $0 < n\epsilon < 1$ for all natural numbers $n \in \mathbb{N}$.

Extended Probability

Indeed, in order to treat compound lotteries in decision trees where successive branches can have infinitesimal probabilities, and also to have a satisfactory theory of subjective probability, it seems desirable to allow an extended probability measure π on a measurable state space $(S, \mathcal{S})$ with σ-algebra $\mathcal{S}$ to take the form of a power series.

Let $>_L$ denote the lexicographic strict ordering defined by

$$\sum_{k=0}^{\infty} a_k \epsilon^k >_L 0$$
$$\iff \exists m \in \mathbb{Z}_+ : a_m = 0 \text{ and } k < m \implies a_k = 0$$
$$\iff \exists e_0 \in \mathbb{R}_+ : e \in \mathbb{R} \cap (0, e_0) \implies \sum_{k=0}^{\infty} a_k e^k > 0$$

Extended Probability Measure

That is, there should be a mapping

$$\mathcal{S} \ni E \mapsto \pi(E) = \sum_{k=0}^{\infty} \pi_k(E)\epsilon^k \in R^*(\epsilon)$$

that satisfies:

1. $\pi(E) >_L 0$;
2. $\pi(S) = 1$;
3. if the countable collection of sets E_n $(n \in \mathbb{N})$ is pairwise disjoint, then $\pi(\cup_n E_n) = \sum_n \pi(E_n)$ (countable additivity).

Extended Expected Utility

At least for the case when S is finite, axioms can be given which imply that a preference ordering $\succsim$ over any combination of roulette and horse lotteries can be represented by the lexicographic weak ordering $\geq_L$ applied to expectations, in the form of a power series in ϵ, of a real von Neumann–Morgenstern utility function v.

Outline

Black Swans

The species *cygnus atratus* was unknown to most of the world before Willem de Vlamingh voyaged to Western Australia in 1697.

John Stuart Mill's classical example in elementary philosophy:

> *"No amount of observations of white swans can allow the inference that all swans are white, but the observation of a single black swan is sufficient to refute that conclusion."*

Black Swan Events

Black swan events, unlike those described in Taleb's book, may not even be imagined *ex ante*.

Even so, their possible effects on the consequences of modelled current decisions can be allowed for, at least in principle, within a suitable EU decision model allowing an "enlivened" version of the usual decision tree.

An Initial Simple Tree

Let Y be a fixed consequence domain.

Assume an agent who wants to maximize the expected value of a von Neumann–Morgenstern utility function $v : Y \to \mathbb{R}$.

Consider a (dead) decision tree

- with an initial (decision) node n_0,
- where the agent chooses one $i \in I$
- that leads to a succeeding chance node n_1^i in $N_{+1}(n_0) = \{n_1^i \mid i \in I\}$,
- where chance determines each possible $j \in J_i$ with a known transition probability $\pi(j|i)$
- that leads to a succeeding terminal node n_2^{ij} in the set $N_{+1}(n_1^i) = \{n_2^{ij} \mid j \in J_i\}$,
- resulting in the final consequence probability distribution $\gamma(n_2^{ij}) \in \Delta(Y)$.

Initial Evaluation

Given the known consequence probability distribution $\gamma(n_2^{ij}) \in \Delta(Y)$, the expected utility of reaching any terminal node n_2^{ij} $(i \in I, j \in J_i)$ is

$$w_2(n_2^{ij}) = \sum_{y \in Y} \gamma(n_2^{ij})(y)v(y) = \mathbb{E}_{\gamma(n_2^{ij})} v(y)$$

,

Working backwards, the expected utility of reaching any preceding chance node $n_1^i \in N_{+1}(n_0)$ is $w_1(n_1^i) = \sum_{j \in J_i} \pi(j|i) w_2(n_2^{ij})$.

An optimal decision maximizes $w_1(n_1^i)$ with respect to $i \in I$, or equivalently, subject to $n_1^i \in N_{+1}(n_0)$.

An Evolving Simple Tree

The agent can hardly make an unmodelled decision,
so we assume that $N_{+1}(n_0)$ remains fixed.

We also assume that any enrichment of the tree
is assumed to take place
after a chosen decision node $n_1^i \in N_{+1}(n_0)$ has been reached.

For each $i \in I$, contingent on reaching n_1^i,
we postulate an expanded set
of succeeding terminal nodes $N_{+1}^+(n_1^i) = \{n_2^{ij} \mid j \in J_i^+\}$
(where $J_i^+ \supseteq J_i$).
' We also have:

- revised transition probabilities $\pi^+(j|i)$;
- revised consequence probabilities $\gamma^+(n_2^{ij}) \in \Delta(Y)$.

Revised Evaluation

Of course, the revised expected utility
of reaching any terminal node n_2^{ij} ($i \in I$, $j \in J_i^+$) is

$$w_2^+(n_2^{ij}) = \sum_{y \in Y} \gamma^+(n_2^{ij})(y) v(y) = \mathbb{E}_{\gamma^+(n_2^{ij})} v(y)$$

The revised expected utility of reaching
any chance node $n_1^i \in N_{+1}(n_0)$
is therefore $w_1^+(n_1^i) = \sum_{j \in J_i^+} \pi^+(j|i) w_2^+(n_2^{ij})$.

Retrospective Evaluation

Ex post, the appropriate decision at initial node n_0
would have been to maximize $w_1^+(n_1^i)$ with respect to $i \in I$.

Ex ante, however, only the details of the original model
can be used, by definition.

What the agent can still do, however,
is to recognize that the original evaluation function $w_1(n_1^i)$
may be revised to an unknown evaluation function $w_1^+(n_1^i)$
that ranges over a function space of possible evaluation functions.

In other words, we can apply a robust decision analysis
and choose the initial decision $i \in I$
in order to maximize $\mathbb{E} w_1^+(n_1^i)$ after allowing for uncertainty
about the appropriate form of the function $i \mapsto w_1^+(n_1^i)$.

Envoi

Grau, teurer Freund, ist alle Theorie;
grün des Lebens gold'ner Baum

Goethe's *Faust*, part I.

Grey, dear friend, is all theory;
green the golden tree of life.

Many thanks for your attention!

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