Notes on Preference Representations for Unknown/Catastrophic/Distant Risks

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Abstract

These are preliminary, incomplete notes toward a survey of models of the behavioral response to catastrophic risks posed by rare, but recurring, natural events. Models of preferences reflecting 'ambiguity aversion' and specific 'fear of catastrophe', and expected utility models with structural uncertainty about the distant future, are introduced.

1 Presentation Outline

1. Introduction
   - What is "catastrophic risk"? Its defining characteristics?
     - Canonical examples: hurricanes, climate change?
       * Hurricanes: regularly recurring, but wide range of damage, with low probability of extreme damage
       * Climate change: single (ongoing) process with an uncertain consequence path, wide variation in consequences over the very long run, and unknown probability of (future) catastrophic damage
   - What is its impact on preferences? How captured in utility representations?
     - Impact of "dread"? Utility unbounded below? 'Certainty equivalent' discontinuity?
     - Perceptions of 'risk'? Fundamental structural uncertainties: future technologies, future rates of return/discounting, ignorance/ambiguity about likelihoods of the 1risks'?
     - Is 'utility' different in a state of disaster?
     - 'Foreseeable' vs. 'unforeseeable' catastrophes?
   - What characteristics of decision models are salient to analyzing 'catastrophic risk'?
     - Properties of utility representations? Impact of distant future, of "end of time"?
     - Impact of 'behavioral anomalies/regularities'?
2. Some Existing Models
   - Basic expected utility model
   - Models with "ambiguity":
     - Choquet expected utility models
     - Rank dependent expected utility and "prospect theory"
     - Multiple prior 'maximin' preferences
     - 'Smooth' models of ambiguity aversion: separating time, risk, an ambiguity preferences
   - Valuation 'sensitive to rare events': the "topology of fear"

- Structural uncertainty about the decision problem in the long run — uncertain, state dependent rates of return

3. How are "catastrophic risks (events)" handled in each of these models?

   - Descriptive vs. Prescriptive uses, relevance
   - Recurring vs. unique 'catastrophes'
   - Impending vs. distant 'catastrophes'

4. Conclusion: What would we like to know to move forward?

2 Some Issues

- Definition of Catastrophic Risk: *possibility of vast loss/damage* — typically understood as 'low probability'
  - Interaction of uncertainty and the temporal dimension; uncertainty/ambiguity regarding both the risks and the structure of the analytic framework (preferences, technologies, rates of return, etc.) increase as the horizon moves farther away, generating/enhancing "thick tails" of the analytic distributions
  - If moderate probability (frequent event), then expected utility with appropriate risk and ambiguity preferences and appropriate discounting provides a decision tool;
    * for prediction of individual behavior must consider individual time preference and 'behavioral' regularities, often violating standard axioms
      · Ignoring, or significantly overestimating ('fear factor'), very 'small' probabilities
      · Underestimating, or accepting as 'certain', probabilities near 1
      · Hyperbolic or other non-exponential (time inconsistent) discounting
    * For public policy analysis and decisions, require intertemporally consistent, equitable, and probabilistically/expectationally 'objective' benefit-cost analysis
      · Incorporate fundamental uncertainty about what is know and what will be known about the 'risk' and its consequences
      · Incorporate fundamental uncertainty about future 'technology' and consequent 'rates of return', even if appropriate pure discount rate is zero
  - If extremely small probability (rare event), the properties of lower tail of distribution become significant for evaluation of 'measurable' events, or non-integral evaluation functionals
    * Thin-tailed distributions (dominated by Gaussian) for measurable events, amenable to standard analysis as above
    * Uncertainty/ambiguity about even thin-tailed distributions can render the *decision-relevant distribution* 'heavy-tailed'
    * Heavy-tailed distributions (power law; stable with $\alpha < 2$; regular variation at $-\infty$) lack finite moments of order above first, and sometimes even the first
      · require direct evaluation of loss risk in tail, related to tail index of regular variation
    * 'Non-measurable' events require a non-integral evaluation functional, e.g. purely finitely additive functional of 'events'

- Agent/DM responses to 'Catastrophic Risk':
  - Can range from ignoring to seriously exaggerating the dangers they pose, particularly when of very low, or unknown, (subjective) probability;

- 'Rational' behavior involves making choices ('acts') maximizing some value (minimizing some loss) given knowledge/beliefs/'fears' about likely consequences of those choices, including 'doing nothing' or 'waiting';
  - Valuations typically modeled in "preference representations" (utility functions) on a space of 'payoff relevant' outcomes;
  - 'beliefs' modeled by bounded real-valued set functions (probabilities, capacities) on a space of payoff relevant 'events';
- Systematic deviations from modelled 'rational behavior' are best understood as a problem with the model — improper modelling of 'preferences' or 'beliefs' — requiring further development or new axiomatizations;
  - If utility or marginal utility *unbounded* as $x\downarrow$, basic expected utility model suffers "tyranny of catastrophic risks"
    - outcomes with 'vanishingly small' probability, but an arbitrarily 'bad' result, dominate choice; the 'certainty equivalent' is the arbitrarily bad outcome $\approx$ unbounded 'willingness to pay to avoid' (Buchholz, Schymura, 2010)
  - If utility or marginal utility *bounded* as $x\downarrow$, basic expected utility model eventually ignores catastrophic risks of arbitrarily small probability (" black swans"), unless 'tail' of catastrophe distribution sufficiently 'thick'.

3 Choquet Expected Utility Model of Ambiguity Aversion (Schmeidler, 1986)

3.1 Notation

(S,Σ) — state space endowed with capacity, $\nu:\Sigma\longrightarrow[0,1]$; $\nu(\emptyset)=0$, $\nu(S)=1$; $\forall A,B\in\Sigma,\ A\subset B\Longrightarrow \nu(A)\le\nu(B)$.

$I_\nu(f)=\int f d\nu\equiv\int_{-\infty}^{0}[\nu(f\ge t)-1]\,dt+\int_0^\infty \nu(f\ge t)\,dt$ — Choquet Integral

3.2 Preference Representation

$V(f)$ — agent evaluation of 'act'/decision f, real-valued, measurable, bounded function:

$$(f\succsim g)\Longleftrightarrow V(f)\ge V(g)$$

$$V(f)\equiv I_\nu(u\circ f)=\int u(f)\,d\nu,$$

hence optimal act, $f^*\in\arg\max V(f)=\max_f I_\nu(u\circ f)$.

3.2.1 Interpretation

- A general formulation of preferences that 'respect' ambiguity (uncertainty about the distribution of 'risks' faced), allowing both 'ambiguity aversion' and 'ambiguity loving';
- Allows uncertainty about likelihoods as is non-additive; Ambiguity aversion generated by *convexity* ('supermodularity') of the capacity ν:
  $\forall A,B\in\Sigma,\ \nu(A\cup B)+\nu(A\cap B)\ge\nu(A)+\nu(B)$.
  - Note: Elsberg has argued that, with truly 'large' ambiguous decisions, decision makers are often "ambiguity embracing"
- In general, $I_\nu(f+g)\neq I_\nu(f)+I_\nu(g)$

- If acts f, g are *co-monotonic* — $\exists p_\pi$, a probability vector for some permutation of states, π, such that $I_\nu(f) = \int_S f dp_\pi = I_\nu(g)$ — then $I_\nu(\cdot)$ becomes additive with respect to f and g; the comparison involves no ambiguity;

- Special case for known 'risks' — RDEU Model (Quiggin, 1982)

3.3 Rank Dependent Expected Utility (RDEU — Quiggin, 1982)

3.3.1 Notation

$i \in \{1, 2, \ldots N\}$ — outcome 'states', orders 'worst' to 'best'

$\overline{x} = (x_1, x_2, \ldots, x_N)$ — payoffs in each outcome state, ordered from lowest to highest payoff

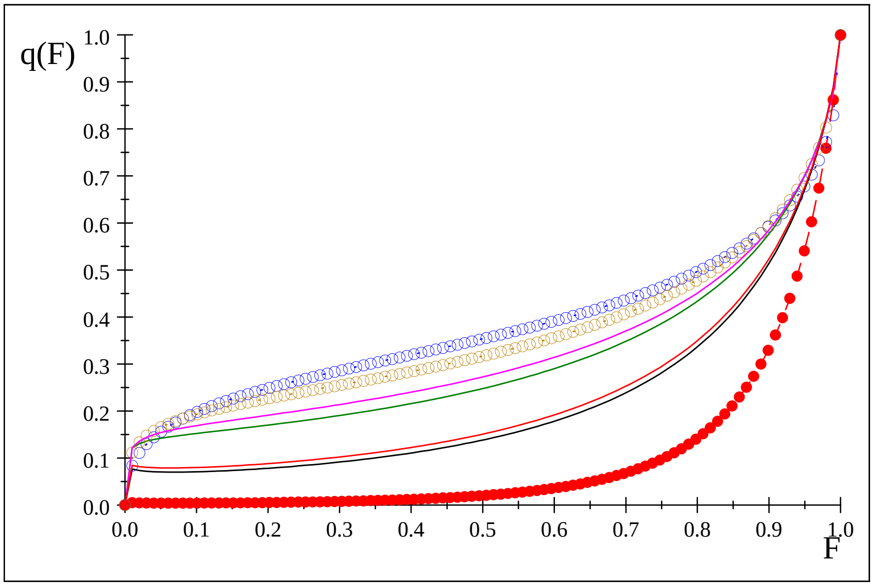
$\overline{p} = (p_1, p_2, \ldots, p_N)$ — probabilities of each 'outcome'; $F(x_i)$ — cumulative distribution function of x, evaluated at x_i

$u : \mathbb{R}^N \longrightarrow \mathbb{R}$ — Bernoulli utility function

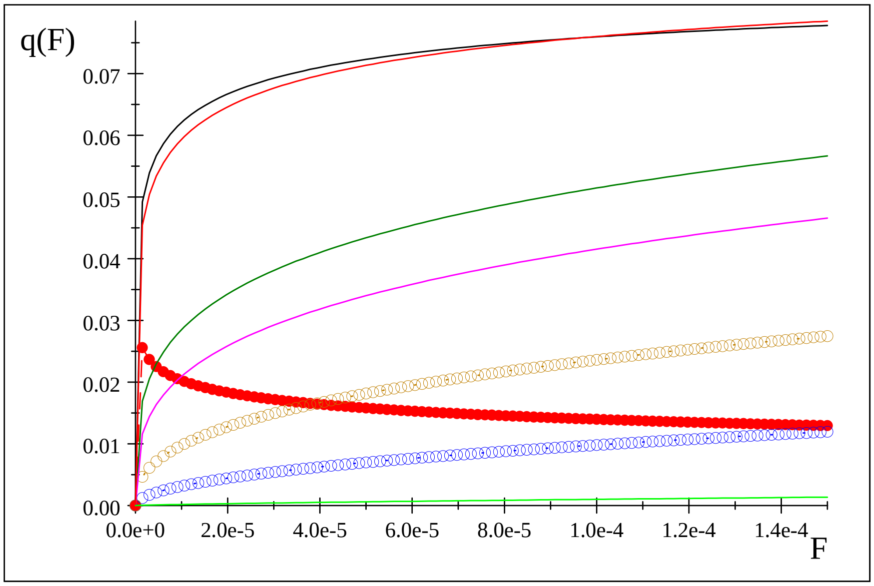
$h_i : \mathbb{R}^N \longrightarrow [0,1]$, $i = 1, 2, \ldots, N$; $h_i(\overline{p}) := q[F(x_i)] - q[F(x_{i-1})]$

$q : [0,1] \longrightarrow [0,1]$, $q(0) = 0$, $q' \geq 0$ — probability weighting function, for example (Quiggin, 1982):

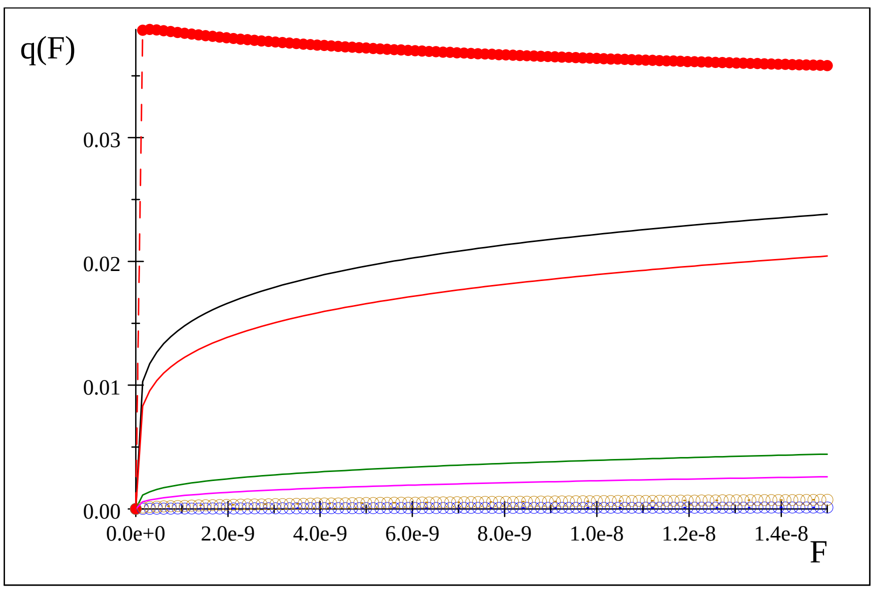
$$q(F) = \frac{F^\gamma}{\left(F^\gamma + (1-F)^{1-\gamma}\right)^{1/\gamma}}$$
$$\gamma \in (0,1)$$



'Rare Event' Probability Perception



'Rare Event' Probability Perception



'Rare Event' Probability Perception

with $\gamma = 0.1$, $q(0.0000000018) = 0.03846$, a 2 billion fold magnification. With $\gamma = 0.2$, $q(0.0011) = 0.082198$, a 75 fold magnification.

3.3.2 Preference Representation

$V : \mathbb{R}^{2N} \longrightarrow \mathbb{R}$ — RDEU function, maximized by agent:

$$V\left(\overline{x}, \overline{p}\right) = \sum_{i=1}^{N} u(x_i) h_i(\overline{p})$$

3.3.3 Interpretation

- Allows non-linear (as revealed in behavioral experiments) probabilities, while preserving first-order stochastic dominance;
- Quiggin's q function 'overweights' extreme events $[h_i(\overline{p}) > p_i,\ i$ near 1 or $N]$ generalizing Tversky, Kahneman (1992);
  - Captures empirical behavioral regularity and both individual and group levels (experimental evidence; Gonzales, Wu, 1999)
  - Overweighting at the bottom may capture 'fear' of the worst outcome, as overweighting at the top may capture 'hope' for the best outcome;
- Concave q implies classic 'risk aversion'.

3.3.4 Example

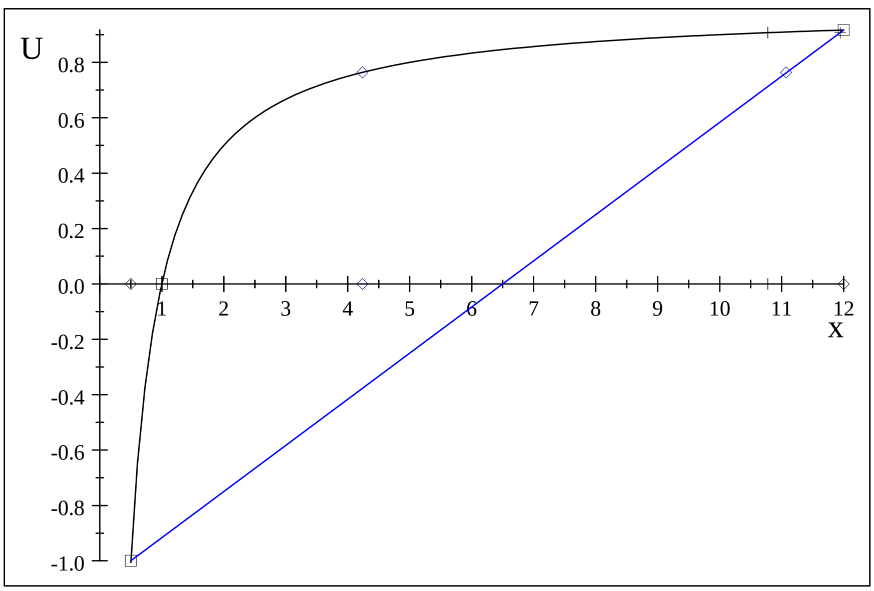
$u(x) = 1 + \frac{x^{1-\eta}}{1-\eta};\ \eta = 2$

$c = 0.5$; $n = 12$ — 'catastrophic' and 'normal' states, with probabilities $(0.005, 0.995)$ respectively

$E(x) = 11.943$; $Eu(x) = 0.907$; Certainty Equivalent is **10.776**

$\gamma = 0.2 \Longrightarrow q(F) = \frac{F^{0.2}}{\left((1.0-1.0F)^{0.8}+F^{0.2}\right)^{5.0}}$; $q\,(0.005) = 0.0795$; $q\,(0.995) = 0.9205$

$RDE(x) = 11.086$; $RDEu(x) = 0.764$; Certainty Equivalent is **4.237**



Objective Catastrophic Risk & RDEU(x) Cert Equiv

3.4 Maximin Expected Utility Model of Uncertainty/Ambiguity Aversion (Gilboa-Schmeidler, 1989)

$\Phi = \{\varphi\}$ — set of potential/conceivable probability distributions over (S, Σ); with general S, Φ assumed weak*-compact

$u\,(\cdot)$ — strictly increasing, continuous, weakly concave

$V(f)$ — agent evaluation of 'act'/decision f, real-valued, measurable, bounded function:

$$(f \succsim g) \Longleftrightarrow V(f) \geq V(g)$$

$$V(f) \equiv \min_{\varphi \in \Phi} \left(\int_S u\left(f\left(s\right)\right) d\varphi\left(s\right) \right),$$

hence optimal act, $f^* \in \arg\max V(f) = \max_f \left\{ \min_{\varphi \in \Phi} \left(\int_S u\left(f\left(s\right)\right) d\varphi\left(s\right) \right) \right\}$.

3.4.1 Interpretation

- Φ captures DMs uncertainty, while $V(\cdot)$ captures 'uncertainty aversion';
- Provides a cognitive interpretation of Choquet Expected Utility, where $\Phi = \mathit{core}\left(\nu\right) = \left\{ \varphi \in \Delta\left(S, \Sigma\right) \middle|\ \varphi\left(A\right) \geq \nu\left(A\right),\ \forall A \subset \Sigma \right\}$, $\Delta\left(S, \Sigma\right)$ — space of all finitely additive probability measures on S;
  - Probabilities can be understood as based on past experience
- Displays strong uncertainty aversion, unwillingness to place second-order distribution over probabilities (as in ζ above)
- If Φ is a singleton, reduces to (subjective) expected utility — there is no ambiguity
- If agent entirely ignorant (unwilling to contemplate) possible probability distributions, reduces to Wald Maximin Criterion: $\max_f \left\{ \min_s u\left(f\left(s\right)\right) \right\}$

4 Smooth 'Ambiguity Aversion': KMM (2005, 2009) – J-PL (2012) Model

4.1 Notation

S — 'states';
D — decisions/outcomes;
$f : S \longrightarrow D$ — 'an act';
$u : D \longrightarrow \mathbb{R}$ — vNM utility;
ζ — subjective probability measure over Θ;
$\forall \theta \in \Theta, \quad \varphi_\theta$ is a probability measure over S;
$v : \mathbb{R} \longrightarrow \mathbb{R}$ — increasing function.

4.2 Preference Representation, Static Model

$$v^{-1}\mathbb{E}_\zeta v\left(u^{-1}\left(\mathbb{E}_{\varphi_\theta} u \circ f\right)\right) \equiv v^{-1}\left(\int_\Theta v\left(u^{-1}\left(\int_S u\left(f\right) d\varphi_\theta \right) \right) d\zeta\left(\theta\right) \right), \tag{1}$$

where $u^{-1}\left(\int_S u\left(f\right) d\varphi_\theta\right)$ is the *certainty equivalent* of the gamble induced by decision f.

$$\left(f \succsim g\right) \Longleftrightarrow v^{-1}\mathbb{E}_\zeta v\left(u^{-1}\left(\mathbb{E}_{\varphi_\theta} u \circ f\right)\right) \geq v^{-1}\mathbb{E}_\zeta v\left(u^{-1}\left(\mathbb{E}_{\varphi_\theta} u \circ g\right)\right) \tag{2}$$

4.2.1 Interpretation

- ζ reflects DM's uncertainty over distribution, φ_θ, governing 'events' in S;
  - formulation separates 'ambiguity' (beliefs) from 'attitude toward ambiguity' (tastes):
    - $|\Theta| > 1$ captures ambiguity;
    - v captures attitude toward ambiguity – concave $\Longleftrightarrow$ 'ambiguity aversion';
      - Ambiguity aversion (KMM): aversion to 'mean preserving spreads' in distribution over $\mathbb{E}_{\varphi_\theta} u \circ f$ induced by ζ and f;

· φ linear: 'ambiguity neutrality', implying reducibility of compound distribution — observationally equivalent to expected utility with subjective prior ζ;

∗ u captures 'attitude toward risk'.

4.3 Preference Representation, Dynamic (Recursive) Model (Ju, Miao, 2011)

4.3.1 Notation

$s^t = (s_1, \ldots, s_t, S, S, \ldots) \in S^\infty$ is DM's information (node s_0 fixed);
$X \equiv (x_t)_{t \geq 0}$ is DM's decision, x_t adapted to s_t and measurable;
Θ — set of states evolving according to Markov chain;
$\varphi_\theta\left(s_{t+1}, s_t \left| s^t\right.\right)$ is distribution over s_{t+1} given history and θ

4.3.2 'Value Function' Representation

$$V_t(X) = W\left(x_t, \mathcal{R}_t\left(V_{t+1}(X)\right)\right) \tag{3}$$

$$\mathcal{R}_t(V_{t+1}) = v^{-1}\left(\mathbb{E}_\zeta\left[v \circ u^{-1} \mathbb{E}_{\varphi_\theta}\left(u\left(V_{t+1}\right)\right)\right]\right) \tag{4}$$

where $V_t(X)$ is the time t continuation value, W is the time aggregator based on current returns and some certainty equivalent of the $t+1$ continuation value.

4.3.3 Interpretation

- $\mathcal{R}_t$ is an 'uncertainty aggregator' mapping $t+1$ continuation value into t certainty equivalent;
  - $v \circ u^{-1}$ linear $\Longrightarrow$ 'ambiguity neutrality' [Epstein-Zin (1989) recursive utility];
  - $v \circ u^{-1}$ non-linear $\Longrightarrow$ aversion to uncertainty about underlying state distribution;
- Aggregator *not indifferent* to timing of resolution of uncertainty:

$$W(x, y) = \left[(1-\beta)\, x^{1-\rho} + \beta y^{1-\rho}\right]^{\frac{1}{1-\rho}}, \ \rho > 0, \rho \neq 1 \tag{5}$$

  with $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$, $v(x) = \frac{x^{1-\eta}}{1-\eta}$, $\gamma, \eta > 0$ and $\neq 1$, where ρ is inverse to elasticity of intertemporal substitution, γ is relative risk aversion parameter, and η is ambiguity aversion parameter. These assumptions yield:

$$V_t(X) = \left[(1-\beta)\, x^{1-\rho} + \beta \left\{\mathcal{R}_t\left(V_{t+1}(X)\right)\right\}^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

$$\mathcal{R}_t\left(V_{t+1}(X)\right) = \left(\mathbb{E}_{\zeta_t}\left[\left(\mathbb{E}_{\varphi_{\theta,t}}\left[V_{t+1}^{1-\gamma}(X)\right]\right)^{\frac{1-\eta}{1-\gamma}}\right]\right)^{\frac{1}{1-\eta}}.$$

  Here $\eta > \gamma \Longleftrightarrow$ ambiguity aversion.

5 Sensitivity to 'Rare'/Extreme Events: Chichilnisky Model (2009)

Based on 'equal treatment' of frequent and rare events, the latter captured by a "topology of fear" generated by the *ess* sup norm, $\|\cdot\|_\infty$ on L_∞.

5.1 Notation

$s \in \mathbb{R}^1$ indexes "states", endowed with $\mu(s)$ Lebesgue measure; assumes underlying 'states' are equiprobable (Lebesgue measure) with the likelihood/probability of events depending on the number of states involved;

$f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are 'acts' generating "lotteries", giving the 'utility' payoffs, $u(f(s))$, $u(g(s))$, to each 'state' s;

$L = L_\infty(\mathbb{R})$ [essentially bounded functions] is the space of 'lotteries';

5.2 Preference Representation

$W : L_\infty(\mathbb{R}) \longrightarrow \mathbb{R}$ is a ranking function representing preferences.

$$W(f) = \lambda \int_{\mathbb{R}} u\left(f(s)\right)\phi_1(s)ds + (1-\lambda)\langle u \circ f, \phi_2\rangle \tag{6}$$

where $\lambda \in (0,1)$, $\phi_1, \phi_2 \in L^*_\infty$ $(= BA(\mathbb{R}))$ are continuous linear functionals on L_∞, $\phi_1 \in L_1(\mathbb{R})$, $\int_{\mathbb{R}} \phi_1(x)dx = 1$, and ϕ_2 is a purely finitely additive measure.

$$(f \succsim g) \iff \lambda \int_{\mathbb{R}} u\left(f(s)\right)\phi_1(s)ds + (1-\lambda)\langle u \circ f, \phi_2\rangle \geq \lambda \int_{\mathbb{R}} u\left(g(s)\right)\phi_1(s)ds + (1-\lambda)\langle u \circ g, \phi_2\rangle$$

Note: Requires $u \circ f$ be a.s. bounded function.

5.2.1 Interpretation

- Forces 'equal treatment' of 'frequent' and 'rare' events, with (preference) λ-weighting on former;
  - How is this formulation 'operationalized'? How is λ identified? How "constructible" is ϕ_2?
- DM attitude toward (extremely) "rare events" is reflected by ϕ_2, which only places a value on events of Lebesgue measure zero, and in λ, the utility weight placed on "normal" outcomes of the 'lottery' f.
  - How is "risk aversion" properly defined here? Is there a concept of "catastrophe aversion" relevant here? Is it separable from risk aversion?
  - How are "heavy tails" seen/demonstrated here?
- In the absence of 'extreme outcomes', $W(f)$ reduces to an expected utility operator.
  - Also occurs with an agent who refuses to consider the possibility of 'extreme events' and hence has preference parameter $\lambda = 1$.
  - In this case, risk attitude is reflected in concavity of u;

5.3 Finite State Space Version

$|S| = S < \infty$; $\{\phi_s\}$ are the (subjective?) probabilities of the states $\{s\} \equiv S$
$f, g \in \mathbb{R}^S$ are 'acts' generating (utility) "lotteries"
$L = \mathbb{R}^S$ — space of lotteries; all topologies equivalent as finite dimensional

5.3.1 A Preference Representation (by analogy with continuous case)

$W : \mathbb{R}^S \longrightarrow \mathbb{R}$ is a ranking function representing preferences.

$$W(f) = \lambda \cdot \langle \phi, f\rangle + (1-\lambda) \cdot \min_s f_s \tag{7}$$

putting extra 'weight' on the catastrophic outcome.

5.3.2 Interpretation

- This appears a 'descriptive' rather than 'prescriptive' formulation; might enter in incentive constraints on policy decisions.
- Following results are argued:
  1. Expected Utility is 'insensitive to rare events'; clear if $|u| < \infty$; Otherwise have "tyranny of catastrophic risks" (Buchholz, Schymura (2010));
  2. $W(f)$ above is 'sensitive to frequent *and* rare events', but is non-linear;

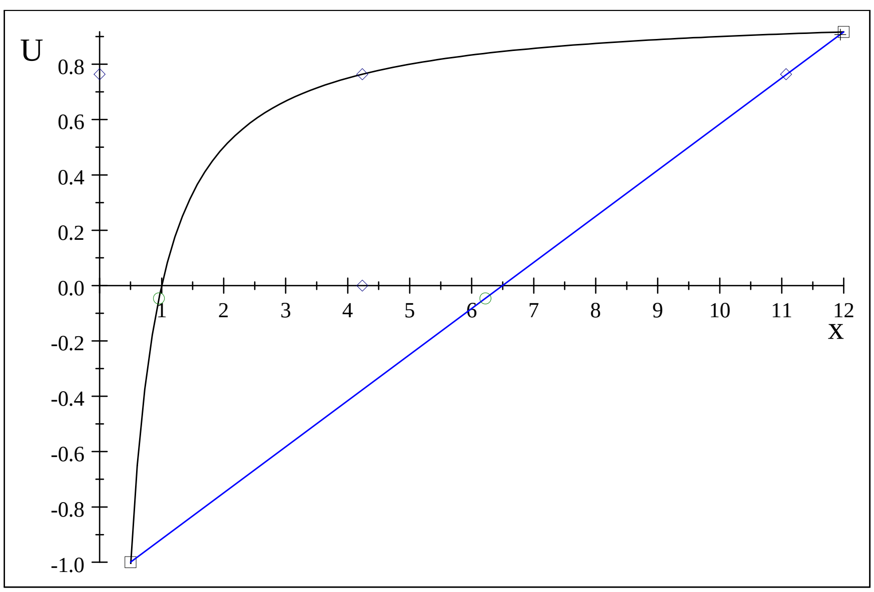
- Assumes that state s^* at which $\min_s f_s$ occurs (the 'catastrophe') is 'rare' — $\varepsilon > \phi_{s^*} > 0$.

- Not clear in Chichilnisky (2010) whether "lotteries" are defined as probabilities over fixed utility payoffs in each state, or as 'utility payoffs' in each state, with its fixed probability.
  - The above formulation assumes lotteries are 'bets', resulting from 'acts', on states with fixed probabilities;
- Depending on λ, can provide a wide range of valuations of lottery with a potential catastrophic outcome, from standard expected utility ($\lambda = 1$) to Wald maximin criterion ($\lambda = 0$);

5.3.3 Example (continued)

Work with the same fixed finite lottery as above, and let λ vary;

- $W(\lambda) = \lambda\left(0.005u(c) + 0.995u(n)\right) + (1-\lambda)\,u\left(c\right) := \lambda \cdot \langle \phi, f \rangle + (1-\lambda) \cdot \min_s f_s$
  - $\Longrightarrow W(\lambda) = 1.90708\lambda - 1.0$
  - $\Longrightarrow W(1) = 0.90708$ and $W(0) = -1.0$
- $\lambda = 0.5 \Longrightarrow W = -0.04646$; $\lambda = 0.92497 \Longrightarrow W = 0.764 = RDEU$ at $\gamma = 0.2$, showing just a little "fear";
- The Certainty Equivalent with $\lambda = 0.5$ is 0.9556, clearly reflecting "fear" of the 'bad' outcome;



'Topology of Fear' & RDEU(x) Cert Equivs

5.3.4 A Dual Version

Here 'acts' map into probabilities on 'states', rather than into payoffs on states.
$|S| = S < \infty$; $\{u_s\}$ are the fixed utility payoffs of the states $\{s\} \equiv S$; $u \in \mathbb{R}^S$
$f, g : S \longrightarrow [0,1] \subset \mathbb{R}^S_+$, are 'acts' generating "lotteries" — probability vectors in unit simplex
$L = \Delta^S$ — space of lotteries; all topologies equivalent as finite dimensional

5.3.5 A Preference Representation (by analogy with continuous case)

$W : \mathbb{R}^S \longrightarrow \mathbb{R}$ is a ranking function representing preferences.

$$W(f) = \lambda \cdot \langle f, u \rangle + (1 - \lambda) \cdot \min_s u_s \qquad (8)$$

putting extra 'weight' on the catastrophic outcome, independent of (subjective) probability of its occurrence.

6 Discounting Expected Utility Models for (Distant) Future Catastrophe

Typically developed in the classic (Subjective) Expected Utility framework;

Argues for using declining and/or 'lowest' discount rates (highest discount factors) for analyzing long run consequences; "fear" of permanent productivity shocks representing 'catastrophic' future states.

6.1 Notation

$\{(r_s, w_s)\}_{s \in S}$ — relevant future 'discount rate' and its probability; or

$f(r)$ — probability density function of (constant) future discount rate

$\Phi(t) = \sum w_s \exp(-r_s t)$ — expected discount *factor* at t

$\exp(-R(t)t) = \Phi(t)$ gives corresponding discount *rate*.

$$\begin{aligned} R(t) &= \frac{-\ln\left(\sum w_s \exp(-r_s t)\right)}{t} \\ &\Longrightarrow \Phi(0) = 1,\ \dot{\Phi}(0) < 0,\ \Phi(\infty) = 0 \\ R(0) &= \sum w_s r_s,\ \dot{R}(t) < 0,\ R(\infty) = \min\{r_s\} \end{aligned}$$

6.2 Base Deterministic Model: Weitzman Formulation of a Standard Model

6.2.1 The Ramsey Model

$y(t) = rK(t) = c(T) + \dot{K}(t)$

$U(c) = \frac{c^{1-\eta}}{1-\eta}$; CRRA $= \eta$; $V(\{c_t\}) = \int_0^\infty \frac{c_t^{1-\eta}}{1-\eta} dt$

$$\Longrightarrow c_r^*(t) = \left[\frac{\eta - 1}{\eta}\right] rK(t) \text{ and } \left[c_r^*(t)\right]^{-\eta} = \left[c_r^*(0)\right]^{-\eta} \exp(-rt) \qquad [FOC] \qquad (9)$$

- Ramsey optimal path is driven by exogenous 'return to capital', r, and η: $g = \frac{r}{\eta}$.

6.2.2 Application: Uncertain Future Return to Capital

Investment opportunity: marginal investment of δ now yields marginal benefit ϵ at time t.

- Critical assumption: $\lim_{c \downarrow 0} U'(c) = +\infty$
  - Marginal cost: $\delta \int_0^\infty \left[c_r^*(0)\right]^{-\eta} f(r) dr$
  - Marginal benefit: $\epsilon \int_0^\infty \left[c_r^*(t)\right]^{-\eta} f(r) dr = \epsilon \int_0^\infty \left[c_r^*(0)\right]^{-\eta} \exp(-rt) f(r) dr$
  - Investing is optimal if $\epsilon \Phi(t) > \delta$, where

Definition 1 *Risk-adjusted Discount Factor is*

$$\Phi(t) \equiv \frac{\int_0^\infty \left[c_r^*(0)\right]^{-\eta} \exp(-rt) f(r) dr}{\int_0^\infty \left[c_r^*(0)\right]^{-\eta} f(r) dr}.$$

The corresponding Discount Rate is

$$R(t) = \frac{-\ln(\Phi(t))}{t}.$$

6.2.3 Gamma Distribution of Future Return to Capital

Assumed to capture the uncertainty about the future interest rate (ex-post the 'driving event', e.g. climate change):

$$f(r) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} r^{\alpha-1} e^{-\beta r}$$

$\Longrightarrow \Phi(t) = \left(\frac{\beta}{\beta+t}\right)^{\alpha-\eta}$ Note that $\alpha = \left(\frac{\mu}{\sigma}\right)^2$ and $\beta = \frac{\mu}{\sigma^2}$. Hence

$$\begin{aligned} \Phi(t) &= \left(\frac{1}{1+\frac{\sigma^2}{\mu}t}\right)^{\left(\frac{\mu}{\sigma}\right)^2-\eta} \\ R(t) &= \frac{\left[\left(\frac{\mu}{\sigma}\right)^2-\eta\right]\ln\left(1+\frac{\sigma^2}{\mu}t\right)}{t} \end{aligned}$$

Notice:

- Sensitivity to η (relative risk aversion), which raises discount factors; $\eta > \frac{\sigma^2}{\mu^2} \Longrightarrow \Phi(t) = 1, R(t) = 0$ $\forall t$.
- Implies substantially higher discounting that exponential due to uncertainty about the rate of return.
- $(\sigma \longrightarrow 0) \Longrightarrow \Phi(t) \longrightarrow \exp(-\mu t)$ and $R(t) \longrightarrow \mu$

6.2.4 The 'Dismal Theorem'

- Weitzman model of climate change: irreversible rise in temperature bringing loss in 'social utility' over the very long run (decision horizon $H \longrightarrow \infty$)
  - Three forms of 'structural uncertainty': (i) how to formulate damages; (ii) how to represent the future path of damaging forces; (iii) how to discount damages.
  - Uses unbounded marginal utility to show *expected* utility losses, driven by randomness in temperature change, $D_E(t) \uparrow \infty$ as $t \to \infty$, giving 'disutility' PDV of $D^*(\delta, H) = \int_0^H D_E(t)e^{-\delta t}dt$, where δ is social 'pure rate of time preference' (arguably near 0).
  - $h(\delta)$ — density function of (subjective) distribution of δ on $[0, a)$, $h(0) = 0$ but $h'(0) > 0$.
  - $D^{**}(H) = \int_0^H D^*(\delta, H)h(\delta)d\delta$

Theorem 2 (Weitzman, 2009) *With "small but non-negligible probabilities,"*

$$\lim_{H\to\infty} D^{**}(H) = \infty.$$

Interpretation

- Derives from above result that with uncertain rates of pure time preference, utilities in far distant future are discounted at lowest possible rate.
- Deals with situations having "potentially unlimited exposure due to structural uncertainty about their potentially open-ended catastrophic reach"
- Rests on 'noticeable' (large) risk-aversion, unbounded marginal utility of consumption at 'zero', and 'thick tails' generated by structural uncertainty

7 'Fat Tails' from Catastrophic Events or 'Structural Uncertainty'

- Power-law tails, reflecting likelihood of 'extreme' events, e.g.
  - Pareto density function: $f(x) = k \cdot x^{-(\beta+1)}$
  - Levy Stable Distributions, with $\alpha < 2$ (infinite variance) or $\alpha < 1$ (undefined mean), with tail distribution:

$$f(x) \sim \frac{c^{\alpha(1+sgn(x)\beta)\sin(\pi\alpha/2)\Gamma(\alpha+1)/\pi}}{|x|^{1/\alpha}}$$

    where α is the 'index', $\beta \in [-1.1]$ is the 'skewness' parameter, a measure of asymmetry, and c is a scale factor.

- Structural uncertainty, generating infinite Moment-Generating function for outcome distribution (Weitzman definition):
  - Long run uncertainty about future value of consumption, c, $c_0 = 1$;
  - Unbounded marginal utility in the face of 'catastrophe': $u(c) = \frac{c^{1-\eta}}{1-\eta} \Longrightarrow u'(c) = c^{-\eta}$
  - Pricing kernel (marginal willingness to pay):

$$M(c) = \beta \frac{u'(c)}{u'(1)} = \beta \exp(-\eta y)$$

    where $y \equiv \ln c$. present consumption willing to give up for a sure unit of future consumption then is:

$$E[M] = \beta E[\exp(-\eta y)] = \beta \int_{-\infty}^{\infty} e^{-\eta x} f(x) dx$$

    where $f(x)$ is the density function of y. Note: $E[M]$ is the moment generating function of the density $f(x)$, where x is a realization of y.

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