A Brief History of Black Swans

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To simplify presentation, only summary definitions and results are provided.

All publications available at www.columbiariskmanagement.net containing definitions and proofs.

Black Swans are Rare Events with Important Consequences

Market Crashes, Natural Hazards: earthquakes, tsunamis, major episodes of extinction, catastrophic climate change

This research is about the foundations of probability when catastrophic events are at stake

It provides a new axiomatic foundation requiring sensitivity to the measurement of both rare and frequent events.

The study culminates with a representation theorem proving existence and representing all probabilities satisfying **three axioms** and specific applications to decision making with black swans

Two axioms include standard notions of continuity and bounded utilities

The third axiom relaxes monotone continuity using a topology used by Debreu - the ‘topology of fear’ - that is appropriate to catastrophes. It requires sensitivity to rare events - a property that is desirable but not satisfied by standard probabilities. These neglect outliers and ”fat tailed” and power law distributions that occur frequently in empirical and experimental observations without theoretical explanation

We explain these axiomatically and include them

We also show the connection between our axioms and von Neumann’s, and the theorems of Godel that are at the foundation of mathematics

Catastrophic Risks are Black Swans

Pentagon's recent report on Climate Change

A recent Pentagon report finds that *climate change* over the next 20 years could result in a global catastrophe costing millions of lives in wars and natural disasters. Potentially most important national security risk

http://www.guardian.co.uk/environment/2004/feb/22/usnews.theobserve
most-commented

http://www.nytimes.com/2009/08/09/science/earth/09climate.html?_r=

http://wwfblogs.org/climate/content/climate-change-climbs-ranks-pentagon-and-cia-0

Our research on the Foundations of Probability with Black Swans provides new foundations for probability, statistics and risk management that improve the measurement, management and mitigation of catastrophic risks.

It updates Mathematical and Economic tools for optimal statistical decisions, providing algorithm for computations and Bayesian updating

New Foundations of Probability

Axioms for *relative likelihoods* or *subjective probability* were introduced more than half a century ago by Villegas, Savage, De Groot, others.

In parallel, Von Neumann and Morgenstern, Hernstein & Milnor, Arrow introduced axioms for *decisions making under uncertainty.*

The two theories are quite different. One focuses on *how things are*, the other on *how we make decisions*.

They are however paralell. Both provide classic tools for measuring and evaluating risks and taking decisions under uncertainty.

US Congress requires such tools for Cost Benefit Analysis of budgetary decisions.

Pentagon focus on extreme cases: security decisions that prevent the worst possible losses.

Show below how classic theories neglect the measurements of extreme situations and rare events with important consequences, the type of catastrophic event that the Pentagon identifies in its recent report.

Evaluations of extreme events, and decisions to prevent and mitigate extreme losses contrast with standard statistical approaches and decisions that use "averages" - weighted by probabilities of occurence.

The purpose of this research is to correct this bias and update existing theory and practice to incorporate the measurement and management of catastrophic risks - focusing on average as well as extremal events.

Traditional Probabilities neglect rare events

Chichilnisky (2010, 2010a) showed that traditional probability and statistics neglect rare events no matter how important their consequences. Based on gaussian or 'normal' distributions (countably additive measures) they exclude 'outliers' and make 'fat tails' and power laws impossible. But these are often observed eg in finance and earth sciences.

Similarly, in decision making, *rationality* is often identified with

Expected Utility Optimization

$$\int_R u(c(t)d\mu(t)$$

For many years experimental and empirical evidence questioned the validity of the *expected utility* model.

Examples are the Allais Paradox, the Equity Premium Puzzle and the Risk Free Premium Puzzle in finance, and the new field of Behavioral Economics.

Discrepancies are most acute when ‘black swans’ or ‘catastrophic risks’ are involved.

This led to adopting "irrational" interpretations of human behavior that seem unwarranted and somewhat unproductive.

Catastrophic Risks are Black Swans

De Finetti followed Dubins & others defined a different foundation of statistics, where subjective probabilities are *finite additive measures*. Controversial, since their distributions give all weight to rare events. Examples Chichilnisky (2010, 2010a).

A middle ground

The new axiomatic foundations we provide for probability and statistics lead to new distributions measuring rare events more realistically than classical statistics. Distributions are neither finitely additive as in De Finetti or Dubins nor countably additive as in De Groot - **they have elements of both**

New Mathematical Developments

for Evaluation and Management of Catastrophic Risks

- *New axioms* for decisions under uncertainty - (Chichilnisky, 1993, 1996, 2000, 2002, 2006, 2009, 2010, 2010a)

- Axioms and results coincide with Von Neumann's in the absence of catastrophic events - theory provides an estension of standrf model - otherwise they are quite different.

- *A new representation theorem* identifies new types of probability distributions.

- Decisions combine standard approaches (which average risk) with distinct reaction to catastrophic or extremal risks (which do not)

- Convex combinations of ‘countably additive’ (*absolutely continuous*) and ‘purely finitely additive’ measures

- Example: Optimize expected returns while minimizing losses in a catastrophe

- A natural decision criterion - but *inconsistent* with expected utility and standard statistical theory.

- Finding new types of subjective probabilities that are consistent with experimental evidence, a combination of finite and countably additive measures

New Results

- New theory appears to agree with experimental and empirical evidence (Chichilnisky and Chanel, 2009, 2010)

- Extends classic theory to problems with catastrophic events (2009-20120

- Creates new Mathematical Results and Tools in Topology, Measure Theory, Functional Analysis, and Stochastic Processes called "Jump - Diffusion" processes (2012).

- Change the way we practice and teach Statistics, Decisions under Uncertainty, Stochastic Processes, Bayesian Analysis Risk Managent and Financial Economics.

Summary of Publications & Applications

- Time: Infinite Horizons & Sustainable Development (1993, 1996, 2000)
- Uncertainty: Choices with Catastrophic Risks (2000, 2002, 2009, 2010, 2012)
- Econometrics: 'NP Estimation in Hilbert Spaces: The Limits of Econometrics' (2006, 2009)
- Neuroeconomics: 'The Topology of Fear' (2009) and
- Experimental Results: Choices with Fear and the Value of Life (2009 and 2011 with Olivier Chanel)
- Survival & Human Freedom: Godel & Axiom of Choice (2007 - 12)

- Green Economics: Climate Change: (2008, 2011)

- New Foundations for Statistics: "The Foundation of Probability with Black Swans" (2010) and "The Foundation of Statistics with Black Swans" (2010)

- Catastrophic Risks with Finite or Infinite States (2011)

- Bayesian Updating with Extremal Risks (2011)

- Axioms explain Jump-Diffusion Stochastic Processes (2012)

Mathematics of Risk

Uncertainty is represented by a family of sets or events $U = \{U_\alpha\}$ whose union describes the universe $U = \cup\{U_\alpha\}$. The relative frequency or probability of an event is a real number $W(U)$ that measures how likely it is to occur. Additivity means that $W(U_1 \cup U_2) = W(U_1) + W(U_2) =$when $U_1 \cap U_2 = \emptyset$ and $W(\emptyset) = 0$.

Example: A system is in one of several states described by real numbers. For risk management in R, to each state $s \in R$ there is an associated outcome, so that one has $f(s) \in R^N$, $N \geq 1$.

A description of probabilities across all states is called a *lottery* $x(s) : R \to R^N$. The space of all lotteries L is thus a *function space* L. Under uncertainty one ranks lotteries in L.

Von Neumann-Morgenstern's (NM) axioms provided a mathematical formalization of how we rank lotteries.

Optimization according to such a ranking is called *'expected utility maximization'* and defines classic decision making under uncertainty.

Expected Utility

Main result from the VNM axioms is a *representation theorem*.

Theorem: (NM, Arrow, De Groot, Hernstein and Milnor) A ranking over lotteries that satisfies the NM axioms admits a representation by an *integral operator* $W : L \to R$, which has as a 'kernel' a countably additive measure over the set of states, with an integrable density. This is called *expected utility*.

Expected Utility Maximization

The NM representation theorem proves that the ranking of lotteries is given by a function

$$W : L \to R,$$

$$W(x) = \int_{s \in R} u(x(s)) d\mu(s)$$

where the real line R is the state space, $x : R \to R^N$ is a "lottery", $u : R^N \to R^+$ is a (bounded) "utility function" describing the utility provided by the (certain) outcome of the lottery in each state s, $u(x)$, and where $d\mu(s)$ is a standard countably additive measure over states s in R.

Ranking Lotteries

Relative likelihoods rankk events by how likely they are to occur. To choose among risky outcomes, we rank lotteries. An event lottery x is ranked above another y if and only if W assigns to x a larger real number:

$$x \succ y \Leftrightarrow W(x) > W(y)$$

where W satisfies

$$W(x) = \int_{s \in R} u(x(s)) d\mu(s)$$

The optimization of an *expected utility* W is a widely used procedure for evaluating choices under uncertainty.

What are Catastrophic Risks?

A catastrophic risk is a small probability (or *rare*) event which can lead to major and widespread losses.

Classic methods do not work:

We have shown (1993, 1996, 2000, 2002) that using NM criteria undervalues catastrophic risks and conflicts with the observed evidence of how humans evaluate such risks.

Problem with NM Axioms

Mathematically the problem is that the measure μ which emerges from the NM representation theorem is countably additive implying that any two lotteries $x, y \in L$ are ranked by W quite independently of the utility of the outcome in states whose probabilities are lower than some threshold level $\varepsilon > 0$ depending on x and y.

This means that expected utility maximization is insensitive to small probablity events, no matter how catastrophic these may be.

Problem with NM Axioms

Expected utility is insensitive to rare events.

A ranking W is called Insensitive to Rare Events when

$$W(x) > W(y) \Leftrightarrow W(x') > W(y')$$

if the lotteries x' and y' are obtained by modifying arbitrarily x and y in any set of states $S \subset R$, with an arbitrarily small probability.

Similarly,

Definition 2: A ranking W is called *Insensitive to Frequent Events* when

$$W(x) > W(y) \Leftrightarrow \exists M > 0,\ M = M(x, y) :$$

$$W(x') > W(y')$$

for all x', y' such that

$$x' = x \text{ and } y' = y \text{ a.e. on } A \subset R : \mu(A) > M.$$

Proposition 1: Expected utility is Insensitive to Rare Events.

As defined by NM, expected utility W is therefore less well suited for ranking situations with catastrophic risks.

Space of lotteries is L_∞ with the sup norm.

New Axioms

Axiom 1. The ranking of lotteries $W : L_\infty \to R$ is sensitive to rare events.

Axioms 2. The ranking W is sensitive to frequent events

Axiom 3: The ranking W is continous and linear

Axioms 2 and 3 are standard, they are satisfied for example by expected utility

Axiom 1 is different and is not satisfied by expected utility. It is a wekening of the standard 'monotone continuity' that requires that the measure of a 'vanishing' family of sets goes to zero. With Axiom 3, for example, *the measure of the family (n,∞)* as n $\to \infty$ *may or not go to zero*

Topology holds the Key

Mathematically, NM axioms postulate nearby responses to nearby events, where

Nearby is measured by *averaging* distances.

In catatrophic risks, we assume the same but we measure distances by *extremals*.

Mathematically the difference is as follows:

Average distance - the L_p norm ($p < \infty$) (and Sobolev spaces)

$$\| f - g \|_p = (\int | (f-g)^p | dt)^{1/p}$$

Extremal distance - the sup. norm of L_∞ :

$$\| f - g \|_\infty = ess \sup_R | (f-g) |$$

Changing the topology, namely the way we measure distances, changes our approach to risk.

It leads to new ways to evaluate risk.

It yields r*egular* measures combined with *singular* measures

Deep Mathematical Roots

The construction of functions to represent singular measures is equivalent to Godel's theorem on the incompleteness of Mathematics and related to the Axiom of Choice and Hahn Banach's theorem

Thus extreme responses to risk conjure up Godel's theorem, the 'Axiom of Choice' and create new types of probability distributions that are both regular and singular, never used before.

Surprisingly, the *sup norm topology* was already used by Gerard Debreu in 1953, to prove Adam Smith's *Invisible Hand Theorem*.

The practical implications of Debreu's results were not clear before. Yet Debreu published his 1953 results in the Proceedings of the US National Academy of Sciences – in an article introduced by Von Neumann himself.

The new axioms have the same status as the second welfare theorem in Economics

Updating Von Neumann Axioms for Choice Under Uncertainty

Axiom 1: Sensitivity to Rare Events

Axiom 2: Sensitivity to Frequent Events

Axiom 3: Linearity and Continuity (in L_∞)

New Axioms

Similarly the new axioms for probability with Black Swans are

Axiom 1: Probabilities are continuous and additive

Axiom 2: Probabilities are unbiased against rare events

Axiom 3: Probabilities are unbiased against frequent events

Axiom 1 *weakens* Arrow's "Axiom of Monotone Continuity" and De Groot's Axiom SP_4, both of which lead to Expected Utility. Indeed:

Theorem 1: The Monotone Continuity Axiom (Arrow, Milnor) is equivalent to "Insensitivity to Rare Events".

Our Axiom 1 is its weakening or logical negation.

Proof: In Theorem 2, "The Topology of Fear", **JME**, 2009 and The Foundations of Probability with Black Swans **JP&S** 2010

A Representation Theorem

Like NM axioms, the new axioms lead to a (new) representation theorem.

Theorem 2 (Chichilnisky 1992, 1996, 2000, 2002)

There exist criteria or functionals $\Psi : L_\infty \to R$ which satisfy all three new axioms. All such functionals are defined by a convex combination of purely and countably additive measures, with both parts present.

Formally, there exists ν, $0 < \nu < 1$, a utility function $u(x) : R \to R$ and a countably additive regular measure μ on R, represented by an L_1 density Γ, such that the ranking of lotteries $\Psi : L_\infty \to R$ is of the form

$$\Psi(x) = \nu \int u(x(s))\Gamma(s)d\mu(s) + (1-\nu)\Phi(u(x(s))).$$

where Φ denotes a purely finite measure on R.

When there are no catastrophic events, the second axiom is void.

Therefore the second component of Ψ "collapses" to its first component, and we have

Theorem 3:

In the absence of catastrophic events, the functional Ψ agrees with VNM's Expected Utility criterion for evaluating lotteries.

New Result

Choices under Uncertainty with Finite States

Theorem 4: A convex combination of Expected Utility and the Maximin criterion satisfies the axioms proposed here.

Proof: Chichilnisky, 20011, related results in Arrow and Hurwicz.

New Result on Limits of Econometrics

Non Parametric Estimation in Hilbert Spaces

with sample space R

Theorem 5: Insensitivy to Rare Events is equivalent to the statistical Assumption SP_4 in Degroot, comparing the relative likelihood of bounded and unbounded events. Both are Necessary and Sufficient for *NP* estimation in Hilbert Spaces on the sample space R.

New Result on Transition to Green Economics (2008)

Renewable Resource Optimization - Survival & Extinction

- Choice with the new Axioms is *equivalent to* optimizing expected utility plus a survival contraint on extinction

- The factor λ that links countable and finite measures, can be identified with the marginal utility of the renewable resource at the point of extinction.

The results extend to how to observe or measure events under uncertainty

- "The Foundations of Probability with Black Swans" - new types of probability distributions (2010)

- "The Foundations of Statistics with Black Swans" - new statistical treatments that are sensitive to catastrophic risks (2011)

Experimental Work & Examples of the new criteria

Finance:

Maximize expected returns while minimizing the drop in a portfolio's value in case of a market downturn

Network optimization:

Electric grids: Maximize expected electricity throughput in the grid, while minimizing the probability of a "black out"

Stochastic Systems:

Jump - Diffussion Processes (eg Merton, 1985).

References

Arrow, K. (1963) *Essays in the Theory of Risk Bearing.*

G. Chichilnisky (2010) "The Foundations of Statistics with Black Swans" *Math. Soc. Sci.* Volume 59, Issue 2, March 2010.

G. Chichilnisky (2010a) "The Foundations of Probability with Black Swans" *J Prob Stat.*, Special Issue on Actuarial and Financial Risks: Models, Statistical Inference, and Case Studies.

G. Chichilnisky "The Topology of Fear" *J. Mathematical Economics*, (2009) 45: 807–816.

G. Chichilnisky "*NP* Estimation in Hilbert Spaces: the Limits of Econometrics" *Econometric Theory*, 2009

D. Cass, G. Chichilnisky and H. Wu "Individual Risks and Mutual Insurance" *Econometrica* (1996) 64, No. 2, 333-341.

G. Chichilnisky and H.M. Wu (2006) " General Equilibrium with Endogenous Uncertainty and Default" *J.Mathematical Economics*.

O. Chanel and G. Chichilnisky "The Influence of Fear in Decisions: Experimental Evidence" *J. Risk Uncertainty*, (2009) 39, No. 3.

O. Chanel and G. Chichilnisky "The Value of Life: Theory and Experimental Observations" Ecological Economics, 2012

G. Chichilnisky "Catastrophic risks" *Int. J. Green Economics* (2009) 3, No. 2. pp. 130-141.

G. Chichilnisky and P. Eisenberger "Asteroids: Assessing Catastrophic Risks" LAMETA, Working Paper DR 2009-13, *J of Probability and Statistics*, 2011.

G. Chichilnisky "An Axiomatic Approach to Choice under Uncertainty with Catastrophic Risks" *Resource and Energy Economics,* 2000.

G. Chichilnisky, "Catastrophic Risk" *Encyclopedia of Environmetrics*, 2002.

G. Chichilnisky "An Axiomatic Approach to Sustainable Development" *Soc. Choice and Welfare* (1996) 13:321-257.

G. Chichilnisky "Catastrophic Risks with Finite or Infnite States" *International Journal of Economics nad Statistics* Vol 23, 2011.

G. Heal, *Lectures on Sustainability*, 1996.

DeGroot M. *Optimal Statistical Decisions* 1970.

N. Hernstein and J. Milnor "An Axiomatic Approach to Measurable Utility" *Econometrica* (1953) 21,:291-297

M. Machina "Expected Utility Analysis without the Independent Axiom" *Econometrica* (1982) 50:277-323.