The Foundations of Probability and Statistics with Black Swans

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To simplify presentation, summary definitions and results are provided. Publications available upon request containing definitions and proofs.

Black Swans are Rare Events with Important Consequences:
Market Crashes, Natural Hazards: earthquakes, tsunamis, major episodes of extinction, catastrophic climate change
This research is about the foundations of probability when catastrophic events are at stake
It provides a new axiomatic foundation requiring sensitivity to the measurement of both rare and frequent events
The study culminates with a representation theorem proving existence and representing all probabilities satisfying *three axioms*
The last of the axioms requires sensitivity to rare events a property that is desirable but not satisfied by standard probabilities.
These neglect outliers and "fat tailed" distributions that are the norm rather than the exception in empirical observations
We also show the connection between our

axioms and the Axiom of Choice at the foundation of mathematics

Catastrophic Risks are Black Swans
Pentagon's recent report on Climate Change

A recent Pentagon report finds that *climate change* over the next 20 years could result in a global catastrophe costing millions of lives in wars and natural disasters.

Potentially most important national security risk

http://www.guardian.co.uk/environment/2004/feb/22/usnews.theobserver#att-most-commented

http://www.nytimes.com/2009/08/09/science/earth/09climate.html?_r=2&pagewanted=1

http://wwfblogs.org/climate/content/climate-change-climbs-ranks-pentagon-and-cia-0

Our research on the Foundations of Probability with Black Swans provides new foundations for probability, statistics and risk management that improve the measurement, management and mitigation of catastrophic risks.

It updates Mathematical and Economic tools for probability and optimal statistical decisions.

New Foundations of Probability

Axioms for *relative likelihoods* or *subjective probability* were introduced more than half a century ago by Villegas, Savage, De Groot, others.

In parallel, Von Neumann and Morgenstern, Hernstein & Milnor, Arrow introduced axioms for *decisions making under uncertainty*.

The two theories are quite different. One focuses on *how things are*, the other on *how we make decisions*.

They are however paralell. Both provide classic tools for measuring and evaluating risks and taking decisions under uncertainty.

US Congress requires such tools for Cost Benefit Analysis of budgetary decisions.

Pentagon focus on extreme cases: security decisions that prevent the worst possible losses.

Classic theories neglect the measurements of extreme situations and rare events with important consequences, the type of catastrophic event that the Pentagon identifies in its recent report.
Evaluations of extreme events, and decisions to prevent and mitigate extreme losses contrast with standard statistical approaches and decisions that use "averages" - weighted by probabilities of occurence.
The purpose of this research is to correct this bias and update existing theory and practice to incorporate the measurement and management of catastrophic risks - focusing on average as well as extremal events.

Traditional Probabilities neglect rare events

Chichilnisky (2010, 2010a) show that traditional probability and statistics neglect rare events no matter how important their consequences. Based on gaussian or `normal' distributions (countably additive measures) they exclude 'outliers' and make `fat tails' and power laws impossible. But these are the rule not the exception (e.g. finance, earth sciences)

Similarly, in decision making, *rationality* is often identified with

Expected Utility Optimization

$$\int_R u(c(t)d\mu(t)$$

For many years experimental and empirical evidence questioned the validity of the ***expected utility*** model.

Examples are the Allais Paradox, the Equity Premium Puzzle and the Risk Free Premium Puzzle in finance, and the new field of Behavioral Economics. Discrepancies are most acute when `black swans' or `catastrophic risks' are involved. This led to adopting "irrational" interpretations of human behavior that seem unwarranted and somewhat unproductive.

Catastrophic Risks are Black Swans

Savage (1963) defined a different foundation of statistics, where subjective probabilities are *finite additive measures*. Controversial, since his distributions give all

weight to rare events. Examples
Chichilnisky (2010, 2010a).

A middle ground

The new foundations of probability and statistics we lead to new distributions measuring rare events more realistically than classical statistics. Distributions are neither finitely additive as in Savage nor countably additive as in De Groot - **they have elements of both**

New Mathematical Developments for Evaluation and Management of Catastrophic Risks

- ***New axioms*** for decisions under uncertainty - (Chichilnisky, 1993, 1996, 2000, 2002, 2006, 2009, 2010, 2010a)
- Axioms and results coincide with Von Neumann's in the absence of catastrophic events - otherwise they are quite different
- ***A new representation theorem*** identifies new types of probability distributions.
- Decisions combine standard approaches (which average risk) with distinct reaction to catastrophic or extremal risks (which do not)
- Convex combinations of `countably additive' (***absolutely continuous***) and `purely finitely additive' measures
- Example: Optimize expected returns while minimizing losses in a catastrophe
- A natural decision criterion - but ***inconsistent*** with expected utility and standard statistical theory.
- Finding new types of subjective probabilities that are consistent with experimental evidence, a combination of finite and countably additive measures

New Results

- New theory appears to agree with experimental and empirical evidence
- Extends classic theory to problems with catastrophic events
- Creates new Mathematical Results and Tools in Topology, Measure Theory, Functional Analysis, and Stochastic Processes called "Jump - Diffusion" processes.
- Change the way we practice and teach Statistics, Decisions under Uncertainty, Stochastic Processes, Bayesian Analysis Risk Managent and Financial Economics.

Summary of Publications & Applications

- Time: Infinite Horizons & Sustainable Development (1993, 1996, 2000)
- Uncertainty: Choices with Catastrophic Risks (2000, 2002)
- Econometrics: `NP Estimation in Hilbert Spaces: The Limits of Econometrics' (2006, 2008)
- Neuroeconomic Theory: `Topology of Fear' (2009)
- Experimental Results: Choices with Fear (2007 and 2009, with Olivier Chanel)
- Survival & Human Freedom: Godel & Axiom of Choice (2007 - 8)
- Green Economics: Climate Change: (2008)
- New Foundations for Statistics: "The Foundation of Probability with Black Swans" (2010) and "The Foundation of Statistics with Black Swans" (2010)

Mathematics of Risk

Uncertainty is represented by a family of sets or events $U = \{U_\alpha\}$ whose union describes the universe $U = \cup\{U_\alpha\}$. The relative frequency or probability of an event is a real number $W(U)$ that measures how likely it is to occur. Additivity means that $W(U_1 \cup U_2) = W(U_1) + W(U_2) =$ when $U_1 \cap U_2 = \emptyset$ and $W(\emptyset) = 0.$

Example: A system is in one of several states described by real numbers. For risk management in R , to each state $s \in R$ there is an associated outcome, so that one has $f(s) \in R^N,$ $N \geq 1$.

A description of probabilities across all states is called a *lottery* $x(s) : R \to R^N.$

The space of all lotteries L is therefore a ***function space*** L. Under uncertainty one ranks lotteries in L .

Von Neumann-Morgenstern's (NM) axioms provided a mathematical formalization of how to rank lotteries.

Optimization according to such a ranking is called ***`expected utility maximization'*** and defines classic decision making under uncertainty.

Expected Utility

Main result from the VNM axioms is a *representation theorem*.

Theorem: (VNM, Arrow, De Groot, Hernstein and Milnor) A ranking over lotteries which satisfies the VNM axioms admits a representation by an *integral operator* $W : L \to R,$ which has as a `kernel' a countably additive measure over the set of states, with an integrable density. This is called ***expected utility***.

Expected Utility Maximization

The VNM representation theorem proves that the ranking of lotteries is given by a function $W : L \to R,$

$$W(x) = \int_{s \in R} u(x(s)) d\mu(s)$$

where the real line R is the state space, $x : R \to R^N$ is a lottery, $u : R^N \to R^+$ is a (bounded) utility function describing the utility provided by the (certain) outcome of the lottery in each state $s,$ $u(x)$, and where $d\mu(s)$ is a standard countably additive measure over states s in $R.$

Ranking Lotteries

Relative likelihoods rankk events by how likely they are to occur. To choose among risky outcomes, we rank lotteries. An event lottery x is ranked above another y if and only if W assigns to x a larger real number:

$$x \succ y \Leftrightarrow W(x) > W(y)$$

where W satisfies

$$W(x) = \int_{s \in R} u(x(s)) d\mu(s)$$

The optimization of an *expected utility* W is a widely used procedure for evaluating choices under uncertainty.

What are Catastrophic Risks?

A catastrophic risk is a small probability (or ***rare***) event which can lead to major and widespread losses.

Classic methods do not work:

We have shown (1993, 1996, 2000, 2002) that using VNM criteria undervalues catastrophic risks and conflicts with the observed evidence of how humans evaluate such risks.

Problem with VNM Axioms

Mathematically the problem is that the measure μ which emerges from the VNM representation theorem is countably additive implying that any two lotteries $x, y \in L$ are ranked by W quite independently of the utility of the outcome in states whose probabilities are lower than some threshold level $\varepsilon > 0$ depending on x and y.

This means that expected utility maximization is insensitive to small probablity events, no matter how catastrophic these may be.

Problem with VNM Axioms

Expected utility is insensitive to rare events.

A ranking W is called Insensitive to Rare Events when

$$W(x) > W(y) \Leftrightarrow W(x') > W(y')$$

if the lotteries x' and y' are obtained by modifying arbitrarily x and y in any set of states $S \subset R$, with an arbitrarily small probability $\cdot$

Similarly,

Definition 2: A ranking W is called *Insensitive to Frequent Events* when

$$W(x) > W(y) \Leftrightarrow \exists M > 0,\ M = M(x,y) :$$

$$W(x') > W(y')$$

for all x', y' such that

$x' = x$ and $y' = y$ a.e. on $A \subset R : \mu(A) > M.$

Proposition 1: Expected utility is Insensitive to Rare Events.

As defined by VNM, expected utility W is therefore less well suited for evaluating catastrophic risks.

Space of lotteries is L_∞ with the sup norm.

New Axioms

Axiom 1. The ranking of lotteries $W : L_\infty \to R$ is sensitive to rare events.

Axioms 2. The ranking W is sensitive to frequent events

Axiom 3: The ranking W is continous and linear

Axioms 2 and 3 are standard, they are satisfied for example by expected utility

Axiom 1 is different and is not satisfied by expected utility.

Topology holds the Key

Mathematically, VNM axioms postulate nearby responses to nearby events, where ***Nearby*** is measured by ***averaging*** distances.

In catatrophic risks, we measure distances by ***extremals***.

Mathematically the difference is as follows:

Average distance - the L_p norm $(p < \infty)$ (and Sobolev spaces)

$$\| f - g \|_p = (\int | (f - g)^p | dt)^{1/p}$$

Extremal distance - the sup. norm of L_∞ :

$$\| f - g \|_\infty = ess \sup_R | (f - g) |$$

Changing the topology, namely the way we measure distances, changes our approach to risk.
It leads to new ways to evaluate risk.
Regular measures combined with *singular* measures

Deep Mathematical Roots

The construction of functions to represent singular measures is equivalent to Hahn Banach's theorem and to the Axiom of Choice.

Thus extreme responses to risk conjure up the `Axiom of Choice' and create new types of probability distributions that are both regular and singular, never used before. Surprisingly, the ***sup norm topology*** was already used by Gerard Debreu in 1953, to prove Adam Smith's ***Invisible Hand Theorem***.

The practical implications of Debreu's results were not clear before. Yet Debreu published his 1953 results in the Proceedings of the US National Academy of Sciences -- in an article introduced by Von Neumann himself.

Updating Von Neumann Axioms for Choice Under Uncertainty

Axiom 1: Sensitivity to Rare Events

Axiom 2: Sensitivity to Frequent Events

Axiom 3: Linearity and Continuity (in L_∞)

Similarly the axioms for probability with Black Swans are
Axiom 1: Probabilities are continuous and additive
Axiom 2: Probabilities are unbiased against rare events
Axiom 3: Probabilities are unbiased against frequent events

Axiom 1 ***negates*** Arrow's "Axiom of Monotone Continuity", which leads to Expected Utility. Indeed:

Theorem 1: The Monotone Continuity Axiom (Arrow, Milnor) is equivalent to "Insensitivity to Rare Events".

Our Axiom 1 is its logical negation.

Proof: In Theorem 2, "The Topology of Fear", JME, 2009

A Representation Theorem

Like VNM axioms, the new axioms lead to a (new) representation theorem.

Theorem 2 (Chichilnisky 1992, 1996, 2000, 2002)

There exist criteria or functionals $\Psi : L_\infty \to R$ which satisfy all three new axioms. All such functionals are defined by a convex combination of purely and countably additive measures, with both parts present.

Formally, there exists ν, $0 < \nu < 1$, a utility function $u(x) : R \to R$ and a countably additive regular measure μ on R, represented by an L_1 density Γ, such that the ranking of lotteries $\Psi : L_\infty \to R$ is of the form

$$\Psi(x) = \nu \int u(x(s))\Gamma(s)d\mu(s) + (1 - \nu)\Phi(u(x(s))).$$

where Φ denotes a purely finite measure on R.

When there are no catastrophic events, the second axiom is void.

Therefore the second component of Ψ "collapses" to its first component, and we have

Theorem 3:
In the absence of catastrophic events, the functional Ψ agrees with VNM's Expected Utility criterion for evaluating lotteries.

New Result

Choices under Uncertainty with Finite States

Theorem 4: A convex combination of Expected Utility and the Maximin criterion satisfies the axioms proposed here.
Proof: Chichilnisky, 2007 see also Arrow and Hurwicz.

New Result on Limits of Econometrics

Non Parametric Estimation in Hilbert Spaces

with sample space R

Theorem 5: Insensitivy to Rare Events is equivalent to the statistical Assumption SP_4 in Degroot, comparing the relative likelihood of bounded and unbounded events. Both are Necessary and Sufficient for ***NP*** estimation in Hilbert Spaces on the sample space R.

New Result on Transition to Green Economics (2008)
Renewable Resource Optimization - Survival & Extinction

- Choice with the new Axioms is $\mathit{equivalent\ to}$ optimizing expected utility plus a survival contraint on extinction
- The factor λ that links countable and finite measures, can be identified with the marginal utility of the renewable resource at the point of extinction.

The results extend to how to observe or measure events under uncertainty

- "The Foundations of Probability with Black Swans" - new types of probability distributions (2010)
- "The Foundations of Statistics with Black Swans" - new statistical treatments that are sensitive to catastrophic risks

Examples of the new criteria

Finance:

Maximize expected returns while minimizing the drop in a portfolio's value in case of a market downturn

Network optimization:

Electric grids: Maximize expected electricity throughput in the grid, while minimizing the probability of a "black out"

Stochastic Systems:

Jump - Diffussion Processes (Merton,

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