Bliss, Catastrophe, and Rational Policy

Kenneth J. Arrow* and Marcel Priebsch§

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Abstract

Lotteries with infinite expected utility are inconsistent with the axioms of expected utility theory. To rule them out, either the set of permissible lotteries must be restricted (to exclude, at a minimum, "fat-tailed" distributions such as that underlying the St. Petersburg Paradox, and power laws that are popular in models of climate change), or the utility function must be bounded. This note explores the second approach and proposes a number of tractable specifications leading to utility functions that are bounded both from above and below. This property is intimately related to that of increasing relative risk aversion as first hypothesized by Arrow (1965).

1 Introduction

The insight that the value attached by a decision maker to a lottery X is more reasonably represented by its expected utility $\mathbb{E}[u(X)]$ under some concave utility function $u(\cdot)$, rather than its mathematical expectation $\mathbb{E}[X]$, dates back to Cramer's (1728) and Bernoulli's (1738) proposed resolutions to the St. Petersburg Paradox.[1] Cramer proposes the candidate utility function $u(x) = \sqrt{x}$,[2] while Bernoulli arrives at $u(x) = \log(x)$ based on the reasoning

*Department of Economics, Stanford University, arrow@stanford.edu

§Department of Economics, Stanford University, priebsch@stanford.edu

[1]Following Cramer's (1728) description of what has become known as the "St. Petersburg Paradox," if an agent receives a payoff of 2^{n-1} with probability $1/2^n$, $n = \{1, 2, \ldots\}$, the mathematical expectation of this lottery is infinite, though, as Cramer puts it, "no reasonable man would be willing to pay 20 ducats as equivalent." This discrepancy is not so much a logical paradox than a challenge to the notion that expected value is an accurate guide to empirical decision making (Menger, 1934).

[2]Cramer also considers a linear utility function that is somewhat arbitrarily truncated at $\overline{x} = 2^{24}$.

that marginal utility should be inversely proportionate to x. Both functions imply a finite certain equivalent for the St. Petersburg lottery.

However, as was first pointed out by Menger (1934), the St. Petersburg Paradox reappears under any unbounded utility function: If we specify payoffs x_n such that $u(x_n) \geq 2^{n-1}$, with probability $1/2^n$, for $n = \{1, 2, \ldots\}$, we obtain a lottery of infinite expected utility. The existence of a lottery with infinite expected utility is (at least implicitly) ruled out in the prevailing treatments of expected utility theory, as it violates continuity of preferences. In light of Menger's (1934) argument, therefore, under any such theory either the set of permissible lotteries must be restricted, or the utility function must be bounded.[3]

von Neumann and Morgenstern (1947) formally prove the existence of an expected utility representation for preferences over lotteries adhering to certain axioms. It is clear that their continuity axioms (3:B:c) and (3:B:d) do not permit a lottery with infinite expected utility,[4] but since they effectively consider only simple random variables as lotteries, they avoid the St.-Petersburg Paradox without restrictions on the utility function.

Savage (1972, p. 80) explicitly discusses the impermissibility of a lottery with infinite utility. In his treatment, this can only be guaranteed by a bounded utility function.[5] The result is proved formally by Fishburn (1970, p. 206) based on the logic of Menger (1934): If $u(\cdot)$ were unbounded, a lottery with infinite utility could be constructed, but such a lottery would be at odds with Savage's postulates (in particular, his variant of continuity axiom).

The requirement that $u(\cdot)$ be bounded above can be relaxed if moment restrictions are imposed on the set of permissible lotteries (Ryan, 1974; Arrow, 1974; Fishburn, 1976). However, in certain applications, not only are such moment conditions violated, but the relevant restriction is the *lower* bound on utility (which cannot be relaxed in an equally straightforward way). Consider, for instance, models of climate change. The distribution of global warming is commonly specified to have "fat tails" (justified either based on the characteristics of the underlying ecological processes, or as the outcome of a Bayesian inference problem) with sizable probability of catastrophically low future consumption.

[3]Menger (1934) discusses the use of a bounded utility function but does not consider it a satisfactory resolution and instead conjectures that agents tend to underestimate small probabilities.

[4]In particular, suppose $\mathbb{E}[u(X)] = \infty$ and $\mathbb{E}[u(Y)] < \mathbb{E}[u(Z)] < \infty$. Then, $\alpha X + (1-\alpha)Y \succ Z$ for any $\alpha \in (0, 1)$, in direct violation of von Neumann and Morgenstern's axiom (3:B:c).

[5]In fact, this insight comes in form of a footnote in the second edition of his book. In the first edition, he argues that an assumption of bounded utility, while reasonable, "would entail a certain mathematical awkwardness in many practical contexts" (Savage, 1954, p. 80), not realizing, at this point, that his postulates *imply* bounded utility.

When sufficiently fat-tailed uncertainty about climate change is combined with a utility function that is unbounded below, the model predicts that society should be willing to sacrifice all but ε of today's consumption to limit global warning. This is Weitzman's (2009) "Dismal Theorem," an extreme result with unclear policy relevance (Pindyck, 2010).

To summarize the discussion, if the set of distributions admissible as lotteries is unrestricted and the underlying state space infinite, and if a decision maker has preferences over those lotteries adhering to the axioms of expected utility theory, the utility function representing those preferences must be bounded from above and below. Many widely-used utility functions are unbounded, including the constant absolute and constant relative risk aversion utility functions. Depending on the application, their use may be an innocuous simplifying assumption (although it is rigorously justified only under additional restrictions). However, in applications including but not limited to those involving fat-tailed distributions (such as the St. Petersburg Paradox and models of climate change), using an unbounded utility function is not justified by expected utility theory and can lead to unexpected or implausible results.

In this note, we propose and explore two tractable specifications of utility functions that are bounded from above and below, and that can be used in those applications where an unbounded utility function is problematic.

2 Preliminaries

For concreteness, suppose the outcome space we are concerned with is levels of consumption, $C = \mathbb{R}_+$. Let $u : \mathbb{R}_+ \mapsto \mathbb{R}$ be a twice continuously differentiable, strictly increasing, concave utility function. Then the coefficients of absolute and relative risk aversion are defined, respectively, as

$$\rho(c) = -u''(c)/u'(c) \tag{1}$$

$$\eta(c) = -cu''(c)/u'(c) \tag{2}$$

(Arrow, 1965; Pratt, 1964). Arrow (1965) hypothesizes that, as an empirical matter, $\rho(c)$ should be decreasing (DARA) and $\eta(c)$ increasing (IRRA) in c. The former conjecture can be traced back at least to Bernoulli (1738, §3) and is generally accepted as consistent with economic intuition, casual observation, and empirical evidence (see Section 3). The latter hypothesis is more difficult to support either introspectively or empirically, but it turns out to possess theoretical plausibility: For $u(\cdot)$ to remain bounded below, it must be that

$\eta(c) < 1$ in some interval $(0, \bar{c})$. For $u(\cdot)$ to remain bounded above, it must not be the case that $\eta(c) \leq 1$ in some interval $(\underline{c}, \infty)$.[6] Thus, if we are willing to assume that $\eta(c)$ is monotonic, then to be consistent with a bounded utility function, it must be monotonically increasing, with $\eta(0) = \lim_{c\to 0} \eta(c) < 1$ and $\eta(\infty) = \lim_{c\to\infty} \eta(c) > 1$.

As a side note, the value of $\eta(0)$ is related to the well-known Inada (1963) condition $\lim_{c\to 0} u'(c) = \infty$. In particular,

$$\eta(0) > 0 \qquad \Rightarrow \qquad \lim_{c\to 0} u'(c) = \infty$$

although the converse is not true.[7]

To incorporate the DARA property, we need the coefficient of relative risk aversion to increase less than linearly. Formally, DARA is equivalent to $(\eta(c)/c)' \leq 0$, which can be rearranged to give

$$\eta'(c) \leq \frac{\eta(c)}{c} \qquad \Leftrightarrow \qquad \frac{d\eta(c)}{d\log(c)} \leq \eta(c) \qquad \Leftrightarrow \qquad \frac{d\log\eta(c)}{d\log(c)} \leq 1$$

that is, relative risk aversion is *inelastic* with respect to consumption.

3 Empirical Evidence on Risk Aversion

The DARA hypothesis is generally confirmed in empirical studies, while evidence on the IRRA hypothesis is mixed. Results are sensitive to aggregation level (household vs. economy), type of sample (cross section vs. time series), choice variable (e.g., portfolio demand vs. insurance demand), and measures of wealth/consumption.

[6]These conditions are mentioned in Arrow (1965). For formal derivations, see, for example, Suen (2009).

[7]Suppose $\eta(0) > 0$. Then by continuity of $\eta(\cdot)$, there exists an interval $[0, \bar{c}]$ on which $\eta(c) > \varepsilon > 0$. Now, note that

$$\begin{aligned} -\log u'(\bar{c}) + \log u'(\underline{c}) &= \int_{\underline{c}}^{\bar{c}} \rho(c)\, dc \\ &= \int_{\underline{c}}^{\bar{c}} \eta(c)/c\, dc \\ &\geq \varepsilon \int_{\underline{c}}^{\bar{c}} 1/c\, dc \end{aligned}$$

so that $u'(\underline{c}) \geq u'(\bar{c})(\bar{c}/\underline{c})^{\varepsilon}$. Hence, when taking $\underline{c} \to 0$ on both sides, we find $\lim_{c\to 0} u'(c) = \infty$.

To show that the converse is not true, consider a utility function that coincides with $u(c) = c(1-\log(c))$ on some interval $[0, \bar{c})$, where $\bar{c} \leq 1$. Then $u'(c) = -\log(c)$, so $\lim_{c\to 0} u'(c) = \infty$. However, $\eta(c) = -1/\log(c)$, so $\lim_{c\to 0} \eta(c) = 0$. All we can say is that if $\lim_{c\to 0} u'(c) = \infty$ and $\eta(c)$ converges to 0 as $c \to 0$, it must be doing so at a rate more slowly than any power of c.

In cross-sectional studies of demand for risky assets, the implied relative risk aversion has been found to be decreasing (Cohn, Lewellen, Lease, and Schlarbaum, 1975), constant (Friend and Blume, 1975), or increasing (Siegel and Hoban, 1982), depending on the measure of wealth and exact methodology employed. Morin and Suarez (1983) report IRRA at low wealth levels but DRRA at higher levels. Peress (2004) points out that evidence typically interpreted in favor of DRRA, namely richer households investing a larger fraction of their wealth in riskier assets, could also be explained by costly information acquisition, an effect that further confounds the picture.

Chiappori and Paiella (2008) make the important observation that a cross section does not necessarily allow us to infer a functional relationship between relative risk aversion and wealth in the presence of heterogeneity, due to endogeneity: High wealth may be the result of past investment decisions, which in turn are reflective of risk aversion. Using panel data, they are unable to reject constant relative risk aversion. Previous time series studies using aggregate insurance demand data produce a variety of results, ranging from inability to reject CRRA (Szpiro (1982) based on property insurance demand), to an isolated finding of increasing *absolute* risk aversion (Eisenhauer (1997) based on life insurance demand).

Halek and Eisenhauer (2001) emphasize heterogeneity in risk aversion between different demographics. Controlling for various individual-specific characteristics (gender, race, religion, marital status, health, age, education, employment status), they find a quadratic relationship between relative risk aversion and net assets (net worth including housing, sample mean $200k): RRA increases at a decreasing rate, and eventually decreases past a wealth level of $4.4m. On the other hand, they find risk aversion to be initially decreasing in human capital (present value of future earnings, sample mean: $225k), and increasing past $2.1m.

In an experimental setting, Gordon, Paradis, and Rorke (1972) find DARA and IRRA for all but subsistence levels of wealth. Binswanger (1981) confirms the DARA finding but also initially reports DRRA, later correcting this finding to IRRA (Quizon, Binswanger, and Machina, 1984). Levy (1994) reports overall support for the DARA hypothesis but not the IRRA hypothesis. There is considerable variation between subjects. Risk aversion depends on wealth within the experiment, but not significantly on real-life wealth or earnings.

As an added complication, estimates of risk aversion tend to be fairly noisy. In their cross-sectional sample, Halek and Eisenhauer (2001) report a mean estimate of $\eta = 3.73$, with a standard deviation of 24.1. Based on an aggregate model and a time series of annual U.S. stock returns from 1926 to 2002, Tödter (2008) obtains a point estimate of

$\eta = 3.5$, with a 95% confidence interval of 1.4 to 7.1. Studies that are unable to reject constant relative risk aversion may therefore suffer from low power and cannot necessarily be interpreted to deliver convincing evidence against IRRA.

4 Specifying Relative Risk Aversion

As discussed in Section 2, a bounded utility function will be implicitly defined by the differential equation (2) if we specify an increasing function $\eta : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $\eta(0) < 1$ and $\eta(\infty) > 1$, where we think of η as relative risk aversion, or, equivalently, the elasticity of marginal utility with respect to consumption (or the inverse of the intertemporal elasticity of substitution in multiperiod models such as the Ramsey growth model). We will also impose that $\rho(c) = \eta(c)/c$ be decreasing. We propose the following candidate specifications.

4.1 Generalized Hyperbolic Specification

The class of utility functions with hyperbolic absolute risk aversion is characterized by linear risk tolerance (defined as the inverse of absolute risk aversion), see Mossin (1968). The corresponding relative risk aversion function is $\eta(c) = c/(ac+b)$. Special cases are constant absolute risk aversion ($a = 0$), constant relative risk aversion ($b = 0$), and quadratic utility ($a = -1$). Note that except in the CRRA case (which is inconsistent with bounded utility), $\eta(0) = 0$ under the hyperbolic specification. To relax this restriction, we propose the following generalization (which is parameterized in a way that naturally incorporates the IRRA and DARA properties):

$$\eta(c) = \bar{\eta}\left[1 - \frac{(1-\beta)\gamma}{c+\gamma}\right] \tag{3}$$

where $\beta, \gamma > 0$ and $\beta\bar{\eta} < 1 < \bar{\eta} < \infty$. It is easy to show that

$$\begin{aligned} \eta(0) &= \beta\bar{\eta} \\ \eta(\infty) &= \bar{\eta} \\ \eta(\gamma) &= \tfrac{1}{2}\eta(0) + \tfrac{1}{2}\eta(\infty) \end{aligned}$$

$\eta(c)$ is strictly increasing but bounded, and $\eta(c)/c$ is strictly decreasing. Furthermore, it can be verified that

$$u'(c) \propto \left[(c+\gamma)^{1-\beta} c^{\beta}\right]^{-\bar{\eta}}$$

A closed-form expression for utility $u(\cdot)$ involving the Gaussian hypergeometric function exists, but in most applications only expressions for marginal utility and relative risk aversion will be required. Where necessary, numerical evaluation of $u(\cdot)$ will likely prove more practicable.

4.2 Power Specification

Under the generalized hyperbolic specification, $\eta(c)$ is bounded above. In applications where consumption grows without bound, it can therefore be expected to behave asymptotically like the CRRA utility function. Where this is considered undesirable or unreasonable, we propose the following alternative power specification:

$$\eta(c) = \underline{\eta} + \beta c^{\gamma} \tag{4}$$

where $\underline{\eta} \geq 0$, $\beta > 0$ and $\gamma \in (0, 1)$. Under these restrictions, relative risk aversion is increasing (without bound), and absolute risk aversion decreasing. Note CRRA and CARA correspond to limiting cases of this specification.

Under power risk aversion,

$$u'(c) \propto \exp\left(-\frac{\beta c^{\gamma}}{\gamma} - \underline{\eta}\log c\right)$$

and a closed-form expression for utility $u(\cdot)$ involving Euler's gamma function exists.

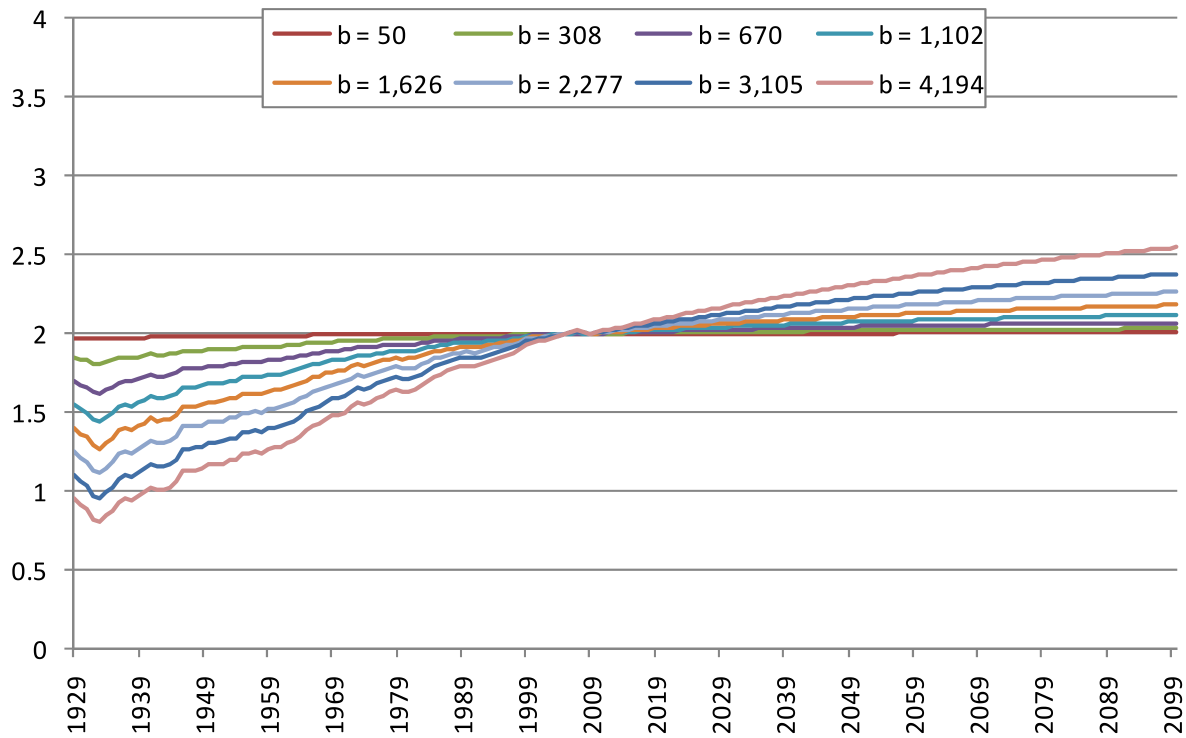
5 Calibration

5.1 Consumption Time Series

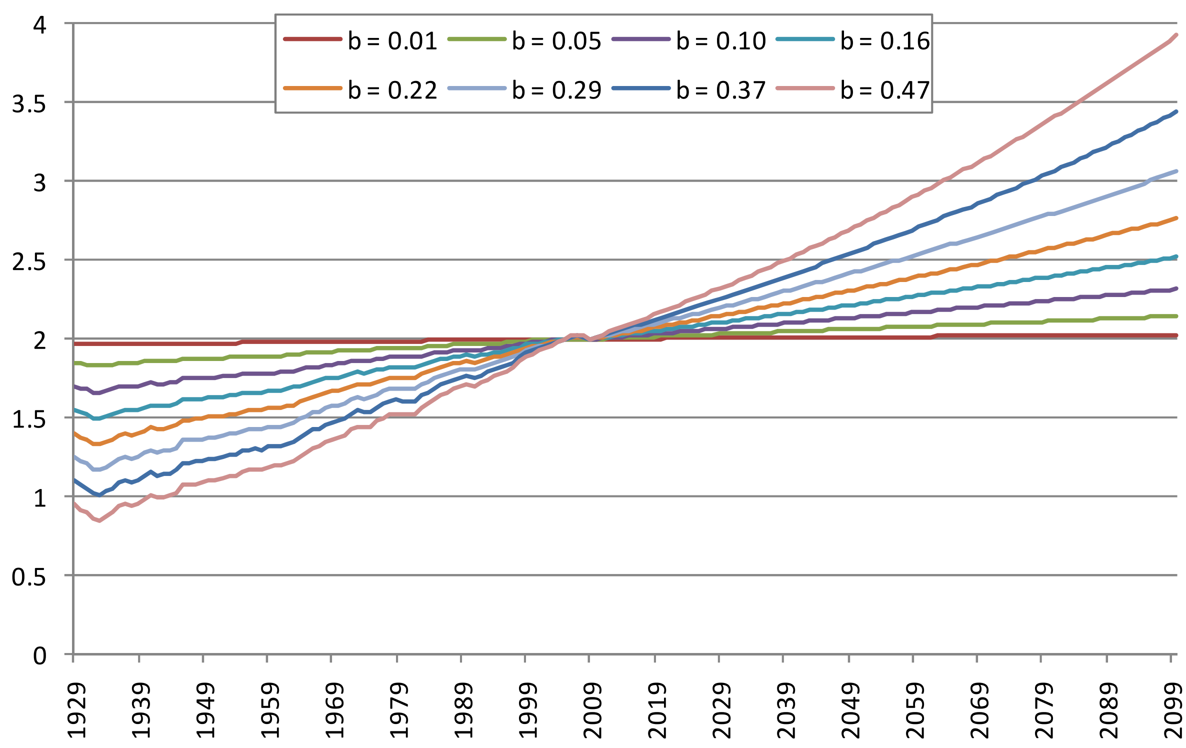
Both generalized hyperbolic and power risk aversion, as given by (3) and (4) respectively, result in a bounded utility function, but under the power specification, $\eta(\cdot)$ itself is unbounded. To develop a better understanding of the quantitative implications, we compute the implied time series of relative risk aversion based on U.S. per capita consumption.

Assuming that the 2009 level of η is 2,Figure 1a shows the evolution of $\eta(c_t)$ over time under several scenarios for specification (3), and Figure 1b shows the same for (4). Future per capita consumption is assumed to grow at an annual rate of 1.6%, consistent with long-term estimates by the Congressional Budget Office.

As Figure 1b shows, if (per capita) consumption grows at an exponential rate, then under the power specification, $\eta(c)$ likewise grows exponentially and without bound. In



(a) $\eta(c_t) = \overline{\eta}[1 - (1 - \beta)\gamma/(c + \gamma)]$



(b) $\eta(c_t) = \underline{\eta} + \beta c_t^{\gamma}$

Figure 1: $\eta(c)$ calibrated to U.S. per capita consumption (future growth assumed at 1.6% p.a.).

contrast, the hyperbolic specification implies that η is bounded above, and Figure 1a reflects this feature.

6 Ramsey Problem

Consider the following Ramsey-style optimization problem (Ramsey, 1928; Cass, 1965; Koopmans, 1965): Find the consumption path to solve

$$\max \int_0^\infty e^{-\delta t} u(C_t)\, dt \tag{5}$$

subject to the capital accumulation equation $\dot{K}_t = f_t(K_t) - C_t$, for given K_0, and with non-negativity constraints on capital and consumption. Then if we denote the (current-value) shadow price of capital by p_t, necessary conditions for an interior solution are

$$u'(C_t) = p_t \tag{6}$$

$$\frac{\dot{p}_t}{p_t} = \delta - f_t'(K_t) \tag{7}$$

In addition, the following transversality condition is commonly imposed:

$$\lim_{t\to\infty} e^{-\delta t} p_t K_t = 0$$

This section explores how the solution to this problem is affected when we replace the popular CRRA utility function by one with generalized hyperbolic or power risk aversion.

To relate the first-order conditions to $\eta(\,\cdot\,)$, differentiate (6) with respect to time, divide by the original equation, and combine with (7), to obtain:

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\eta(C_t)}[f_t'(K_t) - \delta] \tag{8}$$

in addition to which any solution must satisfy the capital accumulation equation

$$\dot{K}_t = f_t(K_t) - C_t \tag{9}$$

Solutions to problem (5) can be found numerically, or analytically as steady state or balanced growth path. Consider the following three scenarios:

6.1 Case I: Cobb-Douglas Production, No Exogenous Growth

Suppose the production function has Cobb-Douglas form with capital intensity $\alpha \in (0, 1)$, and capital depreciates at rate d:

$$f_t(K_t) = AK_t^\alpha - dK_t$$

Then from (8) and (9), the model has a steady state (an optimal path characterized by zero growth in C_t and K_t for appropriate K_0), as follows:

$$K^* = \left(\frac{\alpha A}{d+\delta}\right)^{\frac{1}{1-\alpha}}$$
$$C^* = A(K^*)^\alpha - dK^*$$

Note the steady state does not depend on the utility function, and thus is independent of $\eta(\cdot)$. The transition path and convergence to steady state, on the other hand, will depend on $u(\cdot)$.

To explore whether variation in $\eta(\cdot)$ leads to economically significant differences in transition paths, we can (roughly) calibrate the model to the current state of the U.S. economy under CRRA, generalized hyperbolic, and power risk aversion. Figure 2 plots the (net) production function as well as the policy functions $C(K)$ for all three utility functions, where steady state η is set to 2, and model parameters are chosen to approximately match the current state of the U.S. economy based on national income accounts ($A = 13$, $d = 0.05$, $\alpha = 0.3$). The discount rate is set to $\delta = 0.02$. The power and hyperbolic risk aversion specifications are further calibrated to give $\eta(c_{1929}) \approx 1$, the most extreme scenario in Figures 1a and 1b.

Note the three policy functions almost coincide, with only minor numerical differences in the three specifications. This finding is not unreasonable, since: (1) Under CRRA, the optimal policy $C(K)$ is only mildly sensitive to minor perturbations to η, and (2) for moderate deviations from K^*, $C(K)$ varies within a narrow band by historical standards, so that deviations in K lead to only minor variation in $\eta(C)$ under our calibration assumption that η has not changed dramatically in the last century.

6.2 Case II: Simple Endogenous Ak-Growth

The stationary (zero growth) case is interesting only as a reference point. A simple way to incorporate endogenous consumption growth over time is through a linear production function of the form

$$f_t(K_t) = AK_t - dK_t$$

where $A > d + \delta$. Then (8) implies optimal consumption growth at rate

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\eta(C_t)}[A - d - \delta]$$

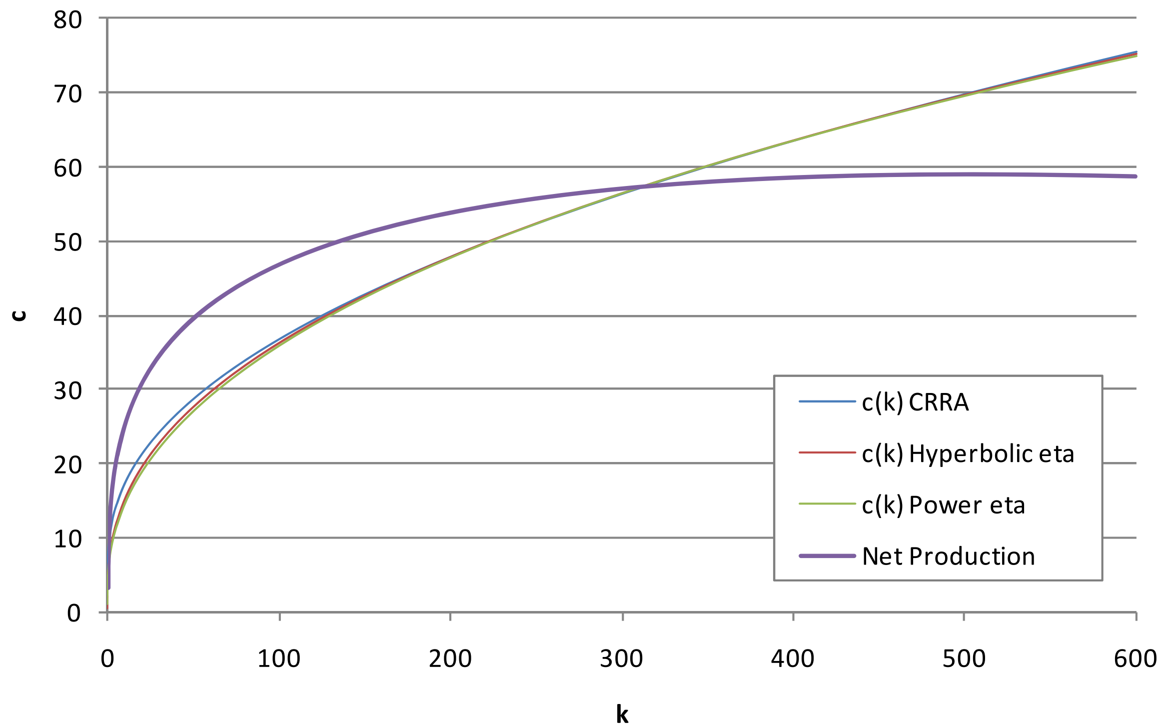


Figure 2: Net production and policy functions

which is constant for CRRA utility, decreasing to an asymptotic level of $\overline{\eta}[A - d - \delta]$ for generalized hyperbolic risk aversion, and decreasing to zero under power risk aversion. Note in the latter case it is only the *rate* of growth that approaches zero – the absolute growth in consumption, $\dot{C}_t$, is in fact increasing over time.

Figure 3 gives us an idea of the magnitudes involved. Assuming a current consumption growth rate of 2%, and current η of 2, it plots the future consumption growth rate for CRRA, generalized hyperbolic, and power risk aversion. The generalized hyperbolic and power risk aversion functions are calibrated so that $\eta(c_{1929}) \approx 1$. Over the next 90 years, consumption growth declines to about 1.5% under generalized hyperbolic risk aversion and to about 1% under power risk aversion. Consumption growth asymptotes to about 1.4% for the hyperbolic specification and to zero for the power specification. Note, however, that even for an horizon of close to a century, the different asymptotic behavior is not immediately obvious from the graph. This suggests a possible challenge in distinguishing between the two forms empirically.

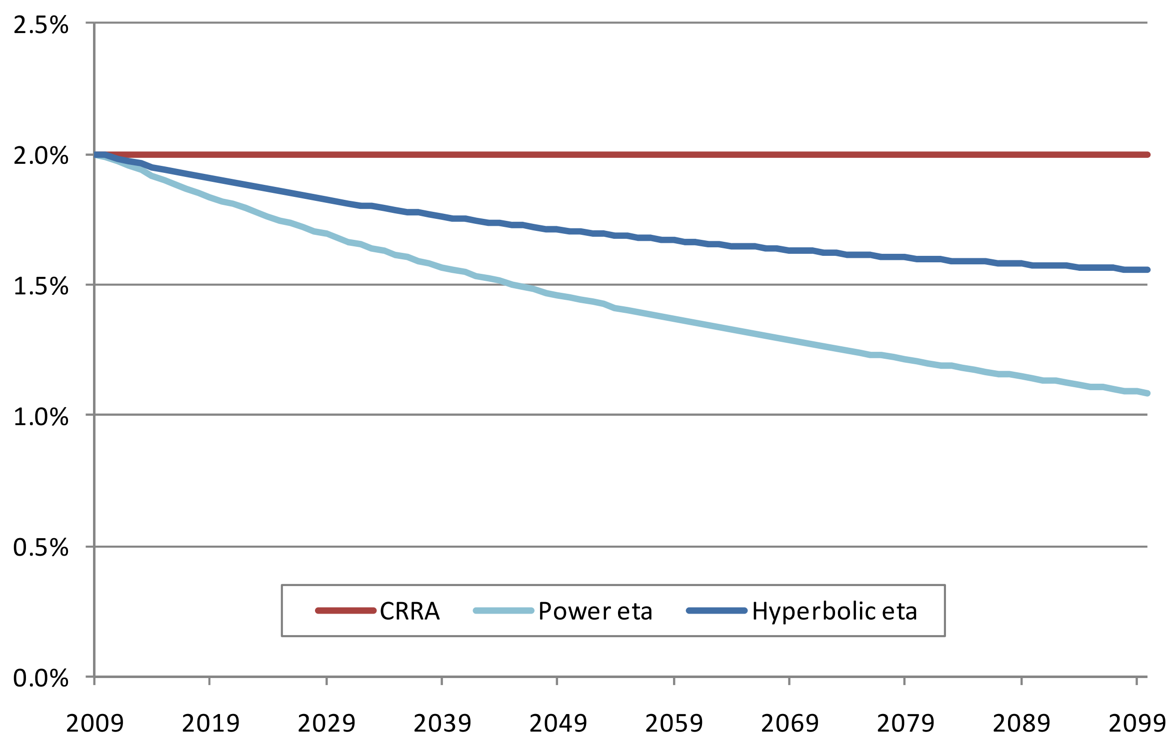


Figure 3: Calibrated per capita consumption growth for Ak-model

6.3 Case III: Cobb-Douglas Production with Exogenous Growth

Suppose now that

$$f_t(K_t) = A_t^{1-\alpha} K_t^{\alpha} - dK_t$$

where $A_t = A_0 \exp(gt)$, that is, technology (implicitly labor-augmenting) grows at a constant exogenous rate $g > 0$. Under CRRA utility, a unit adjustment can be made to show that the steady state from Case I translates into a balanced growth path where

$$\frac{\dot{C}_t}{C_t} = \frac{\dot{K}_t}{K_t} = g$$

This unit adjustment crucially depends on the CRRA assumption, so unlike the steady state we found before, existence and properties of the balanced growth path *do* depend on utility and therefore $\eta(\cdot)$.

6.3.1 Asymptotically Balanced Growth

This section explores asymptotic growth rates of K_t consistent with optimality. With Cobb-Douglas production, we have the following optimality conditions:

$$\frac{\dot{C}}{C} = \frac{1}{\eta(C)}\left[\alpha\left(\frac{A}{K}\right)^{1-\alpha} - d - \delta\right] \tag{10}$$

$$\frac{\dot{K}}{K} = \left(\frac{A}{K}\right)^{1-\alpha} - d - \frac{C}{K} \tag{11}$$

$$\lim_{t\to\infty} e^{-\delta t}u'(C_t)K_t = 0 \qquad \Leftrightarrow \qquad \lim_{t\to\infty} e^{-\int_0^t [\alpha(A/K_t)^{1-\alpha}-d]\,dt}K_t = 0 \tag{12}$$

Consider the following scenarios for the asymptotic behavior of K_t:

1. K_t grows superexponentially. If $\dot{K}/K \to \infty$, then $A/K \to 0$ and hence (11) implies $\dot{K}/K \leq 0$, a contradiction. If $\dot{K}/K \to -\infty$, then $A/K \to \infty$. Hence, from (10), $\dot{C}/C > 0$ eventually. But $Y = A^{1-\alpha}K^{\alpha} \to 0$ superexponentially, so eventually $C > Y$, a contradiction (an immediate contradiction if capital cannot be consumed; if capital can be consumed, the capital stock will reach zero in finite time, at which point $\dot{C}/C > 0$ produces a contradiction).

2. K_t grows subexponentially. If $\dot{K}/K \to 0$, A/K grows at rate g asymptotically, and in particular $A/K \to \infty$. Hence, C must grow at rate $(1-\alpha)g$ to avoid a contradiction in the capital accumulation equation (11). If η is bounded, this presents an immediate contradiction in the Euler equation (10), as $\dot{C}/C \to \infty$. For the Euler equation to be consistent with a constant rate of consumption growth, it must be that η grows at rate $(1-\alpha)g$. But this means $\eta \propto C$ asymptotically, a contradiction (this would require constant absolute risk aversion).

3. K_t grows exponentially. If $\dot{K}/K \to g_K > g$, then $A/K \to 0$, and hence from (11), $\dot{K}/K \leq 0$ in the limit, a contradiction. If $\dot{K}/K \to g_K < g$, then $A/K \to \infty$. For $\dot{K}/K$ to approach a constant, we would necessarily need $\dot{C}/C \to g_C = (1-\alpha)g + \alpha g_K$ (in particular, this also implies $g_C = g_Y$). But for C to grow at constant rate asymptotically, η must grow at rate $(1-\alpha)(g - g_K)$ asymptotically to avoid contradiction in (10). This is not possible for bounded η, but it is possible for the power specification of $\eta(c)$ (since $\dot{\eta}/\eta = \gamma\dot{c}/c$ asymptotically), provided that

$$(1-\alpha)(g - g_K) = g_\eta = \gamma g_C = b[(1-\alpha)g + \alpha g_K]$$

or

$$g_K = \frac{(1-\alpha)(1-\gamma)}{1-\alpha(1-gamma)} g \tag{13}$$

and hence

$$g_C = \frac{1-\alpha}{1-\alpha(1-\gamma)} g \tag{14}$$

Note $0 < g_K < g_C < g$. In this case. Finally, if $\dot{K}/K \to g$, then A/K approaches a constant. From the capital accumulation equation (11), $\dot{C}/C \leq g$, otherwise the consumption capital ratio could not approach a constant. Further, the Euler equation implies that $\eta\dot{C}/C$ approaches a constant, so either η and $\dot{C}/C$ both approach non-zero constants (plausible asymptotic growth path for bounded η), or $\dot{C}/C \to 0$ and $\eta \to \infty$. The latter is a candidate asymptotic growth path for power η but violates transversality condition (12): From the capital accumulation equation, in the limit

$$\alpha\left(\frac{A}{K}\right)^{1-\alpha} - d = g - (1-\alpha)\left(\frac{A}{K}\right)^{1-\alpha} < g$$

and therefore the limit in (12) is infinity, not zero.

To sum up, for bounded η (in particular, the generalized hyperbolic specification), the only candidate asymptotic balanced growth path has $g_K = g_C = g$. Intuitively, since $\eta(C)$ asymptotically approaches $\overline{\eta}$, utility will behave more and more like CRRA on a path with unbounded consumption growth. This suggests an asymptotic balanced growth path that coincides with a CRRA specification where $\eta = \overline{\eta}$.

For power η, the only candidate asymptotic balanced growth path has $g_K < g_C = g_Y < g$, with g_K and g_C given by (13) and (14), respectively. Hence, capital and consumption grow at a rate that falls short of technology growth, and the consumption-capital ratio rises to infinity while the consumption-output ratio is asymptotically constant. The intuitive reason is that as $C_t \to \infty$, the consumer's intertemporal elasticity of substitution decreases ($\eta(C_t) \to \infty$). As a result, immediate consumption is increasingly higher than it would be under a constant elasticity of substitution, at the cost of lower investment and consumption growth.

6.3.2 The Transition Path: Calibration

In the previous section, we discuss asymptotic balanced growth paths under generalized hyperbolic and power risk aversion when the production function is Cobb-Douglas and there is exogenous technology growth. This section calibrates transition paths for a range of plausible parameter values.

The calibrations in this section assume there is population growth at rate n. Exogenous growth rates and technology parameters are set to

$$\delta = 0.02$$
$$\alpha = 0.3$$
$$d = 0.05$$
$$n = 0.015$$
$$g = 0.016$$

Initial capital stock K, labor force L, and technology A are set to their implied 2009 values according to the Economic Report of the President under the following assumptions:

- L_t represents the civilian labor force.
- $Y_t = (A_t L_t)^{1-\alpha} K_t^{\alpha}$ is gross domestic product.
- dK_t is consumption of fixed capital.

Consumption is broadly defined as $Y_t - dK_t$, and per capita variables are computed with respect to the labor force. Finally, lower-case variables (k, c, etc.) denote quantities per technology-adjusted worker (computed by re-normalizing $A_{2009} = 1$ to facilitate interpretation).

In our calibration, we vary the parameters underlying utility specifications (3) and (4). We fix $\eta(c_{2009}) = 2$ and consider different values for $\eta(c_{1929})$ (the earliest year for which detailed national income accounts data are available), and, for the generalized hyperbolic specification, $\eta(\infty)$. While the generalized hyperbolic specification has three free parameters and therefore, in principle, gives us the flexibility to fix the function $\eta(c)$ at three different points, the conditions $0 \leq \beta\overline{\eta} < 1$ impose some restrictions on the relative values $\eta(c)$ can adopt. In particular, given $\eta(c_{2009}) = 2$ and $\eta(\infty)$, the range of permissible $\eta(c_{1929})$ is subject to relatively tight constraints. Figure 4 plots the upper and lower limits of $\eta(c_{1929})$ as a function of $\eta(\infty)$, fixing $\eta(c_{2009}) = 2$. We consider $\eta(\infty) \in \{2.5, 5, 10\}$, and, in light of Figure 4, vary $\eta(c_{1929})$ on a grid that depends on $\eta(\infty)$.

For the power specification, we set $\eta(0) = 0$ and vary $\eta(c_{1929}) \in \{0.5, 1, 1.5\}$, again fixing $\eta(c_{2009}) = 2$.

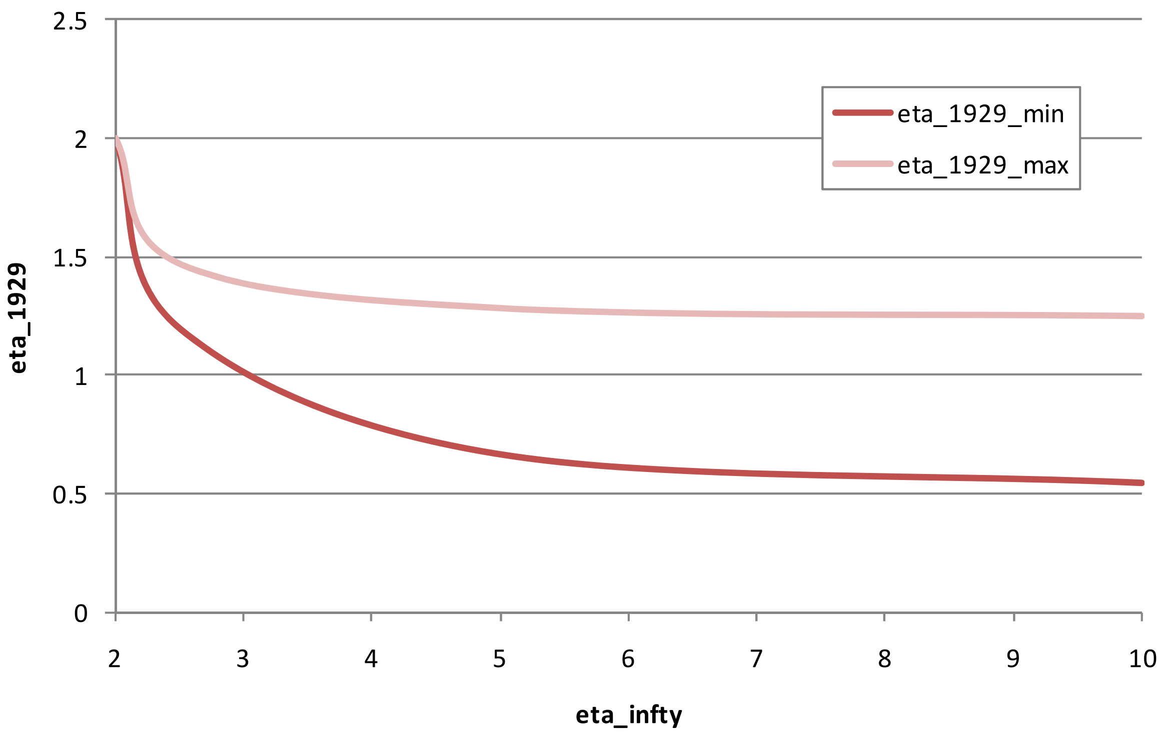


Figure 4: Bounds on $\eta(c_{1929})$ under the generalized hyperbolic specification for given $\eta(\infty)$, fixing $\eta(c_{2009}) = 2$.

Figures 5 through 7 show the transition paths for consumption and capital per technology adjusted worker for generalized hyperbolic risk aversion. The current U.S. capital stock is slightly below the steady state value that would obtain under $\eta = 2$, the hypothesized current level of risk aversion. This is why capital stock and consumption are initially increasing. As consumption grows, $\eta(c)$ increases. Higher η, however, lowers the steady state capital stock, so that capital and consumption eventually decrease, approaching in the limit the asymptotic steady state corresponding to $\eta(\infty)$.

Figure 8 plots the same transition paths for power risk aversion. As before, capital and consumption per technology-adjusted worker are initially increasing for a brief period before gradually declining towards zero. This is because, as discussed above, the asymptotic balanced growth path is characterized by $g_K < g_C < g$.

7 Dismal Theorem

Following Pindyck (2010), consider the following highly simplified model of climate change: Baseline future consumption is given by some value $C_0 > 0$, but effective consumption is reduced by global warming, as follows:

$$C = \frac{C_0}{1 + \lambda T}$$

where $T \geq 0$. For illustrative purposes, we will assume that T follows a Pareto distribution

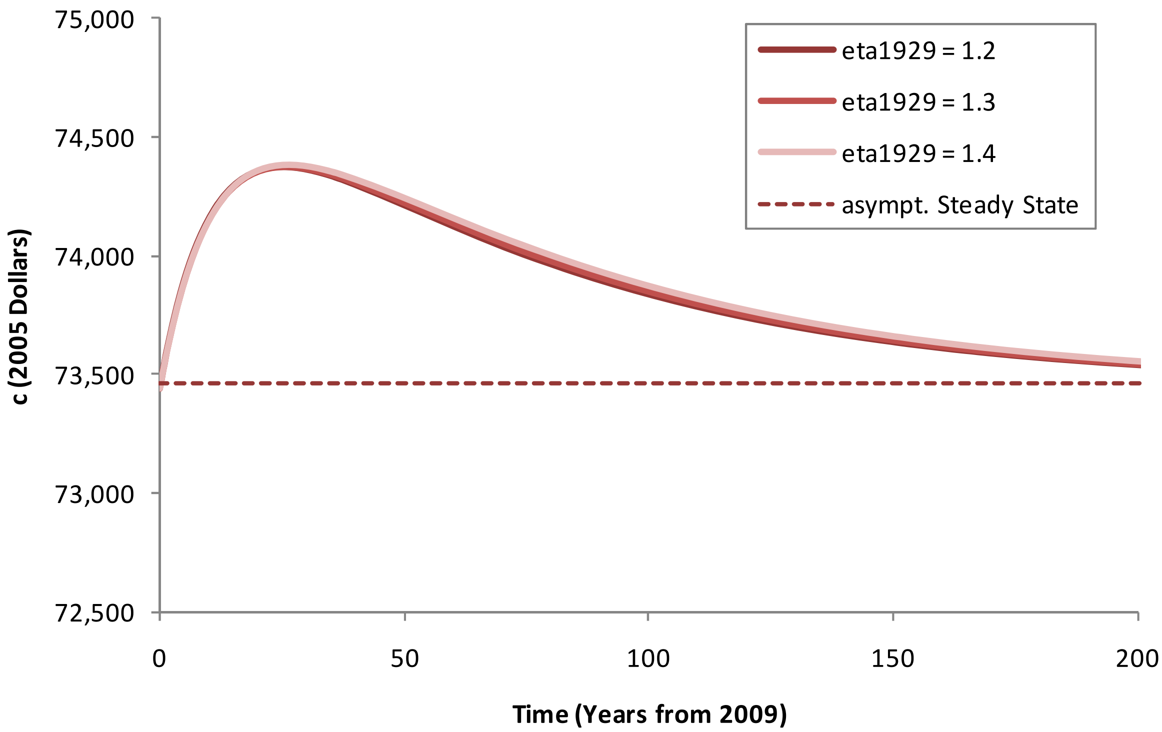
$$f(T) = \theta(1 + T)^{-(1+\theta)}$$

with $\theta = 4/3$. With this parameter, the probability of a temperature increase in excess of 4.5°C is about 10%.[8] Note that $\mathbb{E}(T) = 3$, but higher moments of T do not exist. We will also set $\lambda = 0.2$ so that a temperature increase by 10°C reduces effective consumption by approx. 20%, in line with the projections in Nordhaus (2008).

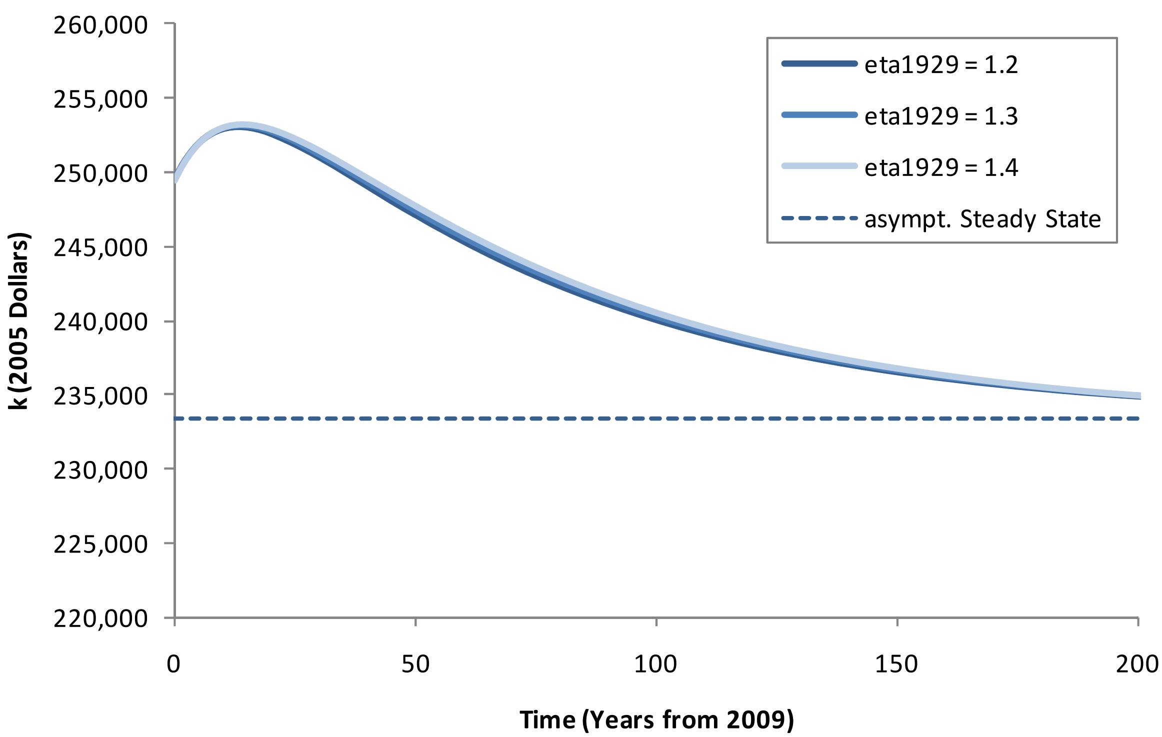
Suppose there is an abatement technology that completely eliminates all global warming. In other words, if this technology is deployed, $T = 0$ almost surely. What fraction of consumption would society be willing to give up to develop this technology?

Normalize baseline future consumption to $C_0 = 1$. If future effective consumption is valued according to a CRRA utility function with $\eta = 2.5$, it is straightforward to verify

[8] An increase of this magnitude corresponds to the upper limit of what the International Panel on Climate Change considers a likely range for the following century.

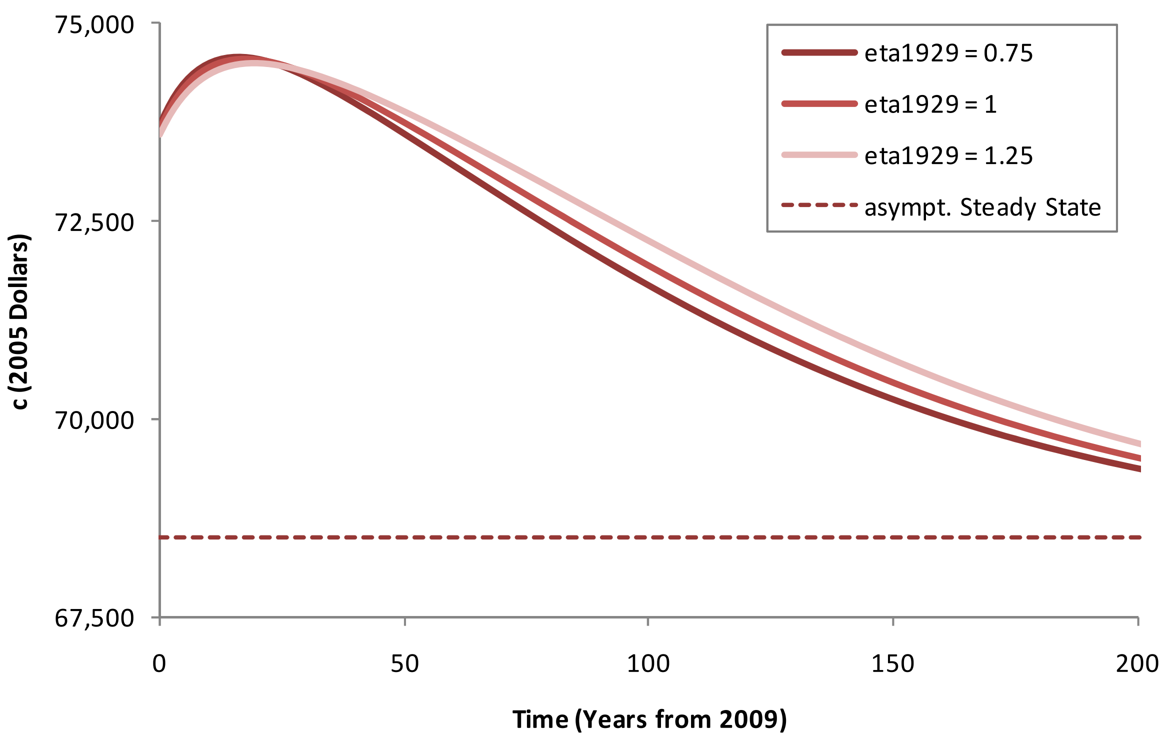


(a) Transition paths for consumption

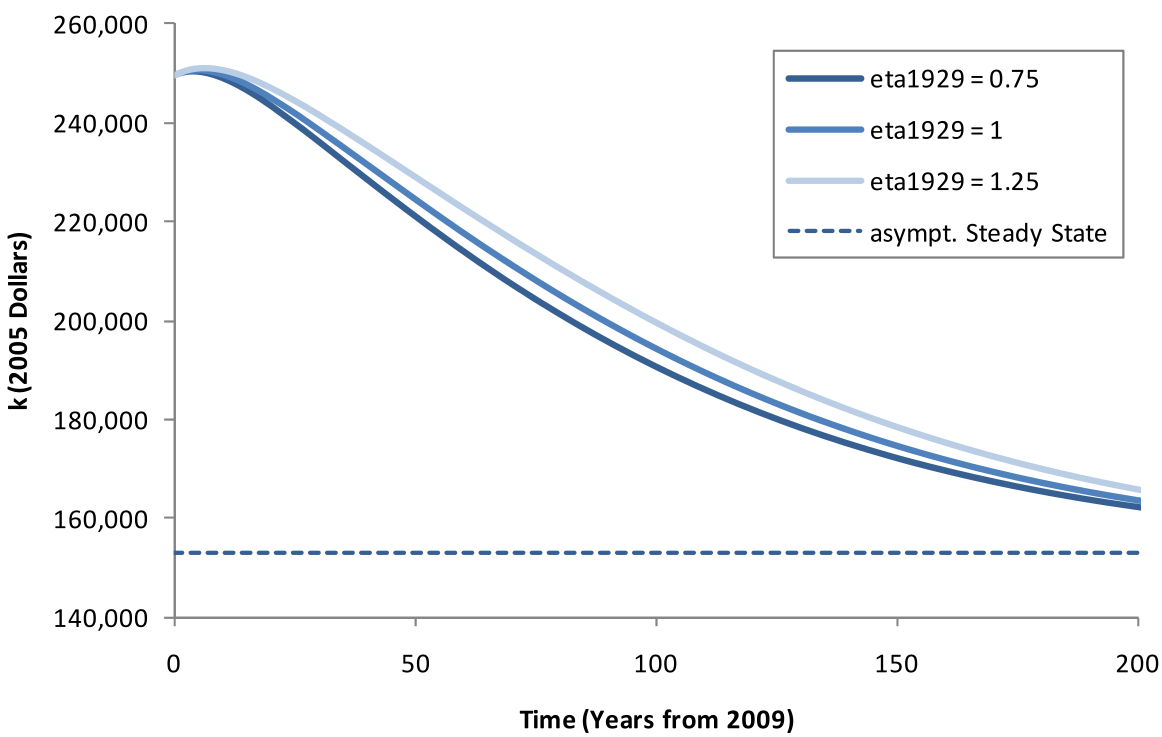


(b) Transition paths for capital

Figure 5: Transition paths based on the generalized hyperbolic specification ($\overline{\eta} = 2.5$).

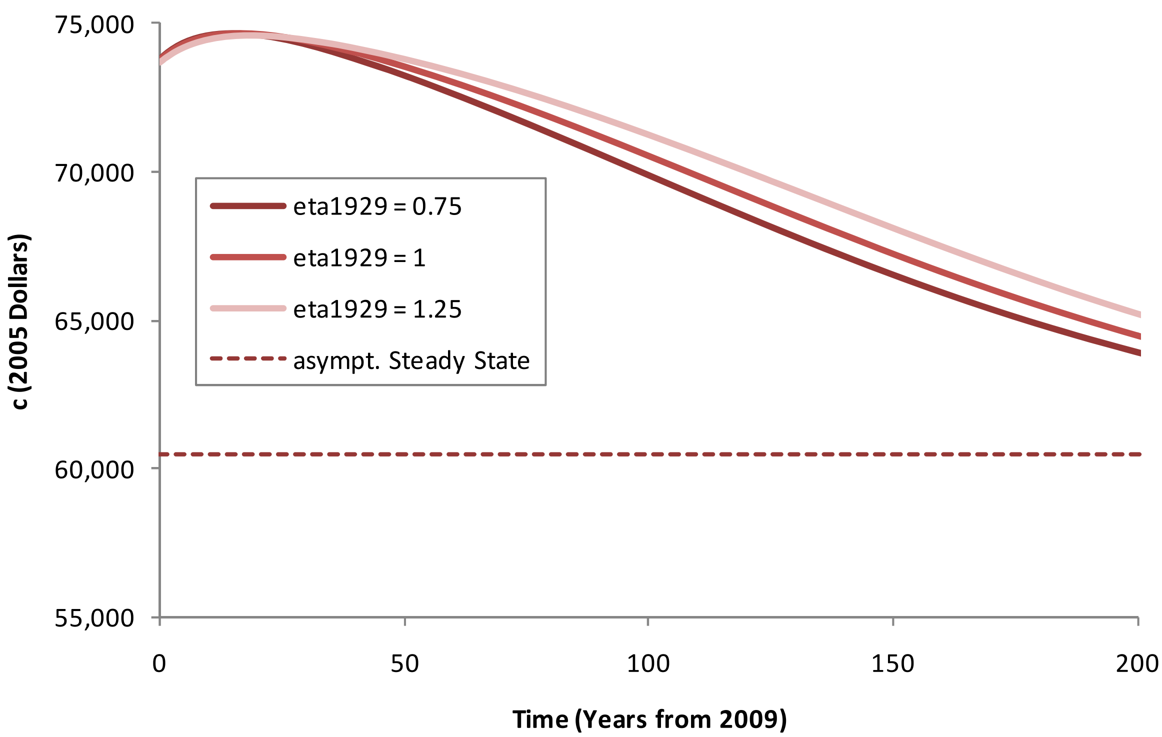


(a) Transition paths for consumption

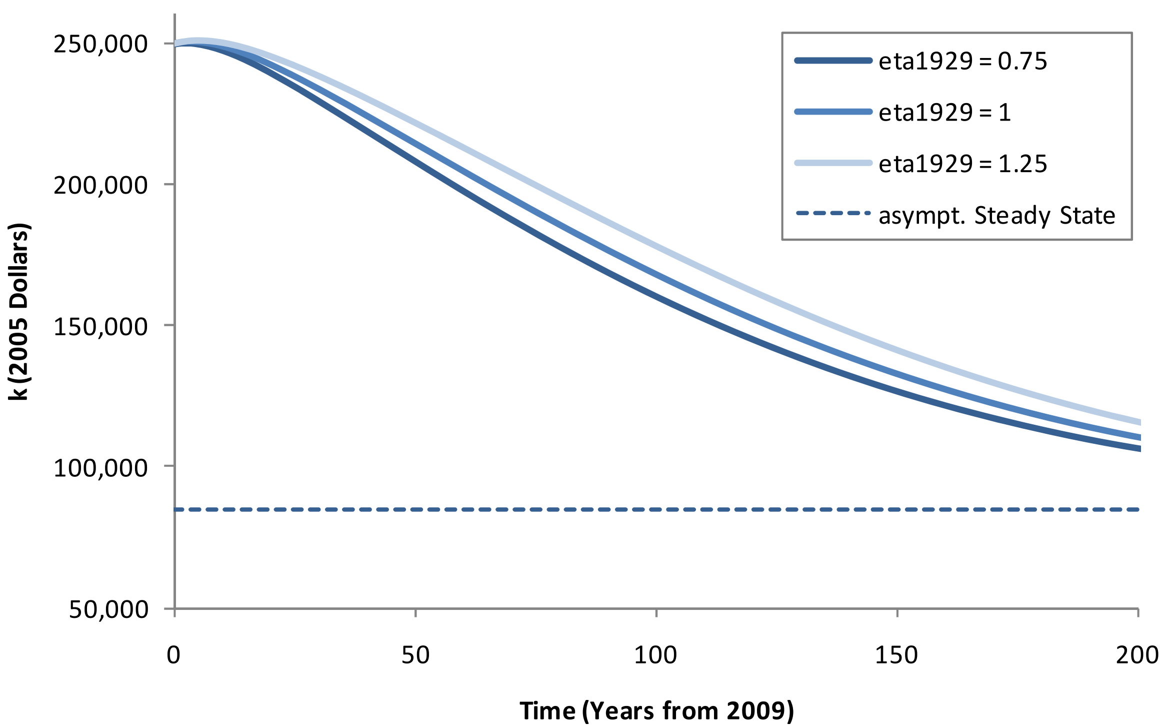


(b) Transition paths for capital

Figure 6: Transition paths based on the generalized hyperbolic specification ($\overline{\eta} = 5$).

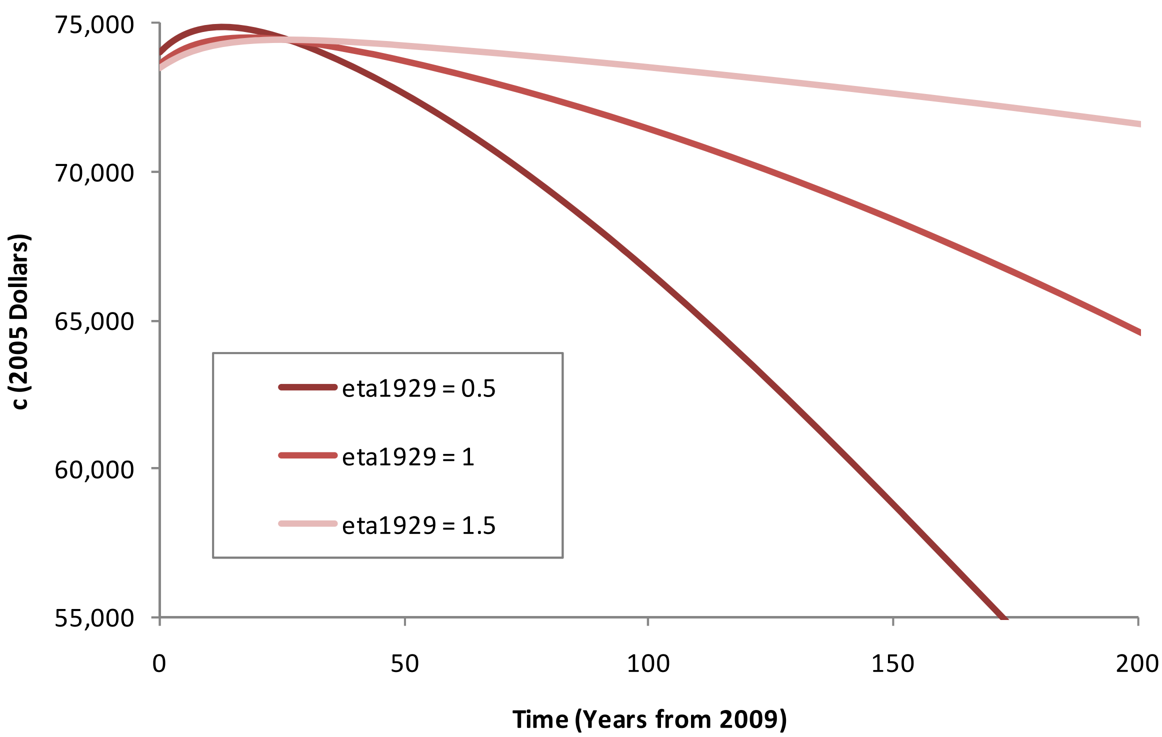


(a) Transition paths for consumption

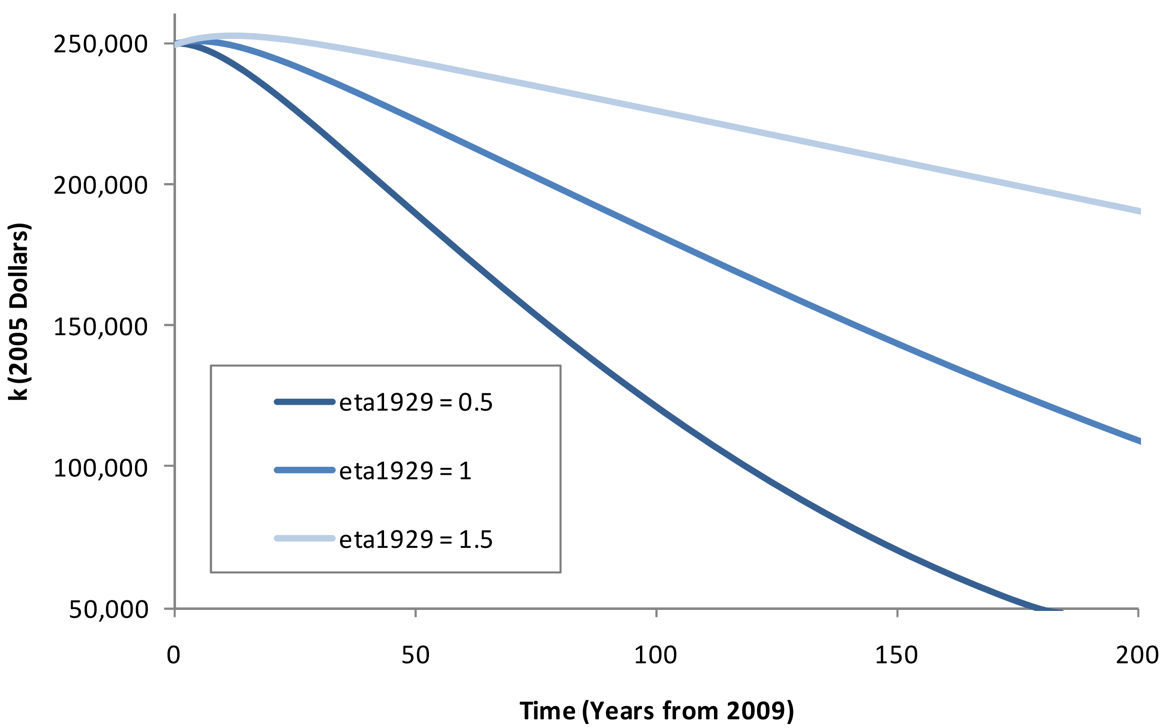


(b) Transition paths for capital

Figure 7: Transition paths based on the generalized hyperbolic specification ($\overline{\eta} = 10$).



(a) Transition paths for consumption



(b) Transition paths for capital

Figure 8: Transition paths based on the power specification ($\underline{\eta} = 0$).

that

$$\mathbb{E}[u(C)] = -\infty$$

At the same time, for any $\varepsilon > 0$,

$$\mathbb{E}[u(\varepsilon C_0)] > -\infty$$

In other words, society should be willing to give up all but a fraction $\varepsilon > 0$ (however small) of future consumption to develop the abatement technology. This is a manifestation of Weitzman's (2009) Dismal Theorem.

On the other hand, suppose we use one of our proposed bounded utility specifications, namely the generalized hyperbolic specification. We similarly calibrate it such that $\eta(1) = 2.5$. Suppose further that $\eta(0) = 0$ and $\eta(\infty) = 5$. Then numerical computations show that

$$\mathbb{E}[u(C)] = \mathbb{E}[u(0.72C_0)]$$

implying society should be willing to give up no more than 28% of future consumption to develop the abatement technology. This is still a sizeable amount, although we should certainly not take the quantitative implications of this over-simplified model too seriously. The important takeaway is that a bounded utility function with identical baseline η leads to a dramatically different *qualitative* result than CRRA utility, namely, that only about a quarter of future consumption should be sacrificed to stop global warming rather than *any* amount of future consumption.

While we chose specific functional forms and parameters values to illustrate this point, the result generalizes as follows: Consider two utility functions $u_1(C, T)$ and $u_2(C, T)$. Suppose $u_1(C, T)$ is unbounded below in T. Further, suppose that $u_2(C, T)$ is uniformly bounded. Then we can always find a distribution of T with the following property: Under u_1, society would be willing to give up *any* fraction $(1 - \varepsilon_1)$ of consumption to deploy the abatement technology, as long as $\varepsilon_1 > 0$, while under u_2, society will be willing to give up no more than $(1 - \varepsilon_2)$, for some $\varepsilon_2 \in (0, 1)$.

8 OLD SECTION: Specification in Terms of Net Interest Rate

As an alternative strategy, we can specify $\eta(\cdot)$ implicitly based on the first-order conditions of a standard optimization problem. Suppose we want to find the optimal consumption path to solve

$$\max \int_0^\infty e^{-\delta t} u(c_t)\, dt \tag{15}$$

subject to the capital accumulation equation $\dot{k}_t = f(k_t) - c_t$, for given k_0 and with non-negativity constraints on capital and consumption. Then if we denote the shadow price of capital by p_t, necessary conditions for an interior solution are

$$\begin{aligned} u'(c_t) &= p_t & \Rightarrow && -\log u'(c_t) &= -\log p_t \\ \frac{\dot{p}_t}{p_t} &= \delta - f'(k_t) & \Rightarrow && \log p_t &= \log p_0 + \int_0^t [\delta - f'(k_t)]\, dt \end{aligned}$$

which, when combined, imply

$$-\log u'(c_t) = -\log p_0 + \int_0^t [f'(k_t) - \delta]\, dt \tag{16}$$

Further, let $\alpha(c) = -u''(c)/u'(c)$ be the coefficient of absolute risk aversion, and $\eta(c) = c\alpha(c)$ the coefficient of relative risk aversion. Then it follows from the Fundamental Theorem of Calculus that

$$-\log u'(c_t) = -\log u'(c_0) + \int_{c_0}^{c_t} \alpha(c)\, dc \tag{17}$$

$$-\log u'(c_t) = -\log u'(c_0) + \int_{\log c_0}^{\log c_t} \eta(c)\, d\log c \tag{18}$$

If we combine (16) with the last pair of equations, we thus obtain

$$-\log u'(c_0) + \int_{c_0}^{c_t} \alpha(c)\, dc = -\log p_0 + \int_0^t [f'(k_t) - \delta]\, dt$$

$$-\log u'(c_0) + \int_{\log c_0}^{\log c_t} \eta(c)\, d\log c = -\log p_0 + \int_0^t [f'(k_t) - \delta]\, dt$$

Let

$$A(c_t) = -\log u'(c_0) + \int_{c_0}^{c_t} \alpha(c)\, dc = -\log u'(c_0) + \int_{\log c_0}^{\log c_t} \eta(c)\, d\log c$$

Then by the Fundamental Theorem of Calculus,

$$\frac{dA}{dc} = \alpha(c) \qquad\qquad \frac{dA}{d\log c} = \eta(c)$$

Our requirements that $\alpha(c)$ be decreasing and $\eta(c)$ increasing thus translate into $A(\cdot)$ being concave in c and convex in $\log c$. Therefore, once we invert the functional relationship between c_t or $\log c_t$ and the cumulative net interest rate $R_t \equiv \int_0^t [f'(k_t) - \delta]\, dt$, we conclude that c_t is an increasing function of R_t, say, $c_t = B(R_t)$, which is *convex* and *logarithmically concave* (an application of the Inverse Function Theorem).[9]

8.1 Characterizing Valid Functions

We can characterize the requirements that $B(\cdot)$ be increasing, convex, and log concave in terms of its derivatives as follows:

$$\frac{dB}{dR} \geq 0 \qquad \frac{d^2B}{dR^2} \geq 0 \qquad \frac{d^2B}{dR^2} \leq \frac{(dB/dR)^2}{B(R)} \tag{19}$$

Alternatively, we can pick an increasing concave function $f(\cdot)$ and let $B(R) = e^{f(R)}$. It remains to ensure convexity of $B(\cdot)$, which amounts to the restriction

$$\frac{d^2B}{dR^2} \geq 0 \qquad \Leftrightarrow \qquad f''(R) \geq -[f'(R)]^2$$

8.1.1 Example

Suppose $f(R) = a + b\log(R)$, with $b > 1$. Then, $f''(R) = -b/R^2 \geq -b^2/R^2$, so that $B(R) = \exp(a + b\log R)$ satisfies all three conditions in (19).

9 OLD SECTION: Relationship Between the Approaches in Section 4 and Section 8

In select cases, we can solve explicitly for the function $B(\cdot)$ from Section 8 corresponding to a given $\eta(\cdot)$.

[9] In addition, we require $d\log B/dR > 1$ for small enough R and $d\log B/dR < 1$ for large enough R, to be consistent with bounded $u(\cdot)$.

9.1 Power Risk Aversion

If

$$\eta(c) = ac^b \tag{20}$$

we can solve (17) or equivalently (18) as

$$-\log u'(c_0) + ac_t^b/b - ac_0^b/b = -\log p_0 + \int_0^t [f'(k_t) - \delta]\, dt$$

The choice of c_0 was arbitrary and by construction, the LHS does not, in fact, depend on c_0. We may therefore set it to a convenient value (say, to eliminate the third term on the LHS by setting $c_0 = 0$ if $b > 0$ and $c_0 = 1$ if $b = 0$), and take the first term to the RHS to normalize p_0 (which depends on the – arbitrary – scaling of marginal utility). A final rearrangement gives

$$c_t = \begin{cases} \left[-\frac{b}{a}\log\tilde{p}_0 + \frac{b}{a}\int_0^t [f'(k_t) - \delta]\, dt\right]^{1/b} & \text{if } b > 0 \\ \exp\left(-\frac{1}{a}\log\tilde{p}_0 + \frac{1}{a}\int_0^t [f'(k_t) - \delta]\, dt\right) & \text{if } b = 0 \end{cases} \tag{21}$$

9.2 Hyperbolic Risk Aversion

If

$$\eta(c) = c/(ac + b) \tag{22}$$

we can solve (17) or equivalently (18) as

$$-\log u'(c_0) + \log(ac_t + b)/a - \log(ac_0 + b)/a = -\log p_0 + \int_0^t [f'(k_t) - \delta]\, dt$$

Setting $c_0 = (1 - b)/a$ and defining $\hat{p}_0$ as normalized price, we can solve for c_t as follows:

$$c_t = -\frac{b}{a} + \exp\left(-a\log\hat{p}_0 + a\int_0^t [f'(k_t) - \delta]\, dt\right)$$

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